

POTENTIALLY CRYSTALLINE DEFORMATION RINGS AND SERRE WEIGHT CONJECTURES: SHAPES AND SHADOWS

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ABSTRACT. We prove the weight part of Serre's conjecture in generic situations for forms of $U(3)$ which are compact at infinity and split at places dividing p as conjectured by [Her09]. We also prove automorphy lifting theorems in dimension three. The key input is an explicit description of tamely potentially crystalline deformation rings with Hodge-Tate weights $(2, 1, 0)$ for K/\mathbb{Q}_p unramified combined with patching techniques. Our results show that the (geometric) Breuil-Mézard conjectures hold for these deformation rings.

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1. INTRODUCTION

Let p be a prime. Serre's modularity conjecture ([Ser72]) predicts that any continuous, irreducible, odd Galois representation $\bar{\rho} : G_{\mathbb{Q}} \stackrel{\text{def}}{=} \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$ is modular. In other words, there exists a cuspidal modular form $f = \sum_{n>0} a_n q^n$, which is an eigenvector for the Hecke operators, such that $\text{tr}(\bar{\rho}(\text{Frob}_{\ell})) \equiv a_{\ell} \pmod{p}$ for all primes $\ell \neq p$ not dividing the level of f . In [Ser87], Serre made his conjecture more precise, by specifying the *minimal weight* (for prime to p level) of such a modular form. More precisely, if $\bar{\rho}$ is modular, then the set of weights in which a modular form f associated to $\bar{\rho}$ occurs is determined in an explicit way from the local datum $\bar{\rho} \stackrel{\text{def}}{=} \bar{\rho}|_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$. This is known as *the weight part* of Serre's conjecture.

In recent years, considerable progress has been made on generalizations of Serre's weight conjecture ([BDJ10], [Sch08], [ADP02], [Her09], [EGH13]...), leading to complete results for 2-dimensional Galois representations ([GK14], [GLS15], [New14]). A key insight in [GK14] connects the weight part of Serre's conjecture to the Breuil-Mézard conjecture ([BM02], [BM14]), and its geometrization ([EG14], [EG]), which predicts the multiplicities of the special fibers of deformation spaces (or, more generally, moduli stacks) of local Galois representations when $\ell = p$ in terms of p -modular representations of general linear groups. In particular, a good understanding of the geometry of local Galois deformation spaces leads naturally to modularity lifting results, Breuil-Mézard and the weight part of Serre's conjecture, via the patching techniques of Kisin-Taylor-Wiles.

In dimension two, potentially Barsotti-Tate (BT) deformation rings were studied via moduli of finite flat group schemes ([Kis09c], [Bre00]) leading to explicit presentations when K/\mathbb{Q}_p is unramified ([BM14], [EGS15]). The geometry of these (potentially) BT-deformation rings is a key input into the proof of the weight part of Serre's conjectures in [GK14] and provided evidence towards mod- p local Langlands. However, a satisfactory understanding of the n -dimensional analogue, potentially crystalline deformation rings with Hodge-Tate weights $(n-1, n-2, \dots, 0)$, seemed out of reach, due to the difficulty of understanding the monodromy operator in the theory of Breuil-Kisin modules.

In this paper, we overcome this difficulty in dimension 3 and achieve a complete description of the local deformation rings $R_{\bar{\rho}}^{(2,1,0),\tau}$ for K/\mathbb{Q}_p unramified and τ a generic tame inertial type. We thereby obtain the first examples in dimension greater than 2 of Galois deformation rings which are neither ordinary nor Fontaine-Laffaille. Our results are consistent with the Breuil-Mézard conjecture and lead to improvements in modularity lifting.

Results on local deformation spaces. Let K/\mathbb{Q}_p be a finite unramified extension. We fix a sufficiently large finite extension E/\mathbb{Q}_p , \mathcal{O} its ring of integers and \mathbb{F} its residue field (the *rings of coefficients* for our representations).

Let $\tau : I_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_3(\mathcal{O})$ be a tame inertial type and $\bar{\rho} : G_K \rightarrow \mathrm{GL}_3(\mathbb{F})$ be a continuous Galois representation. In Definition 3.6, we introduce a notion of *genericity*, which is a mild condition on the inertial weights of $\bar{\rho}$. Our main local results (cf. Corollary 5.14, Theorem 6.9 in the main body of the paper) are a detailed description of framed potentially crystalline deformation ring $R_{\bar{\rho}}^{(2,1,0),\tau}$ in terms of the notion of *shape* attached to the pair $(\bar{\rho}, \tau)$ (cf. Definition 3.3). The *shape* is an element in the Iwahori-Weyl group of GL_3 of length ≤ 4 , and arises from the study of moduli of Kisin modules with descent datum in §2.1 (inspired by work of [Bre14, BM14, CDMb, EGS15] and further pursued in [CL]); it generalizes the notion of *genre* which is crucial in [Bre14] in describing tamely Barsotti-Tate deformation rings for GL_2 .

Theorem 1.1. *Let $\bar{\rho} : G_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_3(\mathbb{F})$. Let τ be strongly generic tame inertial type. Then the framed potentially crystalline deformation ring $R_{\bar{\rho}}^{(2,1,0),\tau}$ with Hodge-Tate weights $(2, 1, 0)$ has connected generic fiber and its special fiber is as predicted by the geometric Breuil-Mézard conjecture.*

If the shape of $(\bar{\rho}, \tau)$ has length at least 2, then $R_{\bar{\rho}}^{(2,1,0),\tau}$ has an explicit presentation given in Table 6. If the shape of $(\bar{\rho}, \tau)$ has length ≤ 1 , then the special fiber of $R_{\bar{\rho}}^{(2,1,0),\tau}$ is described in Section 8.

The first step towards Theorem 1.1 is a detailed study of the moduli space of Kisin modules with descent datum. The shapes of Kisin modules which arise as reductions of potentially crystalline representations with Hodge-Tate weights $(2, 1, 0)$ are indexed by $(2, 1, 0)$ -admissible elements in the Iwahori-Weyl group of GL_3 defined by Kottwitz and Rapoport (cf. [PZ13, (9.17)]). For generic τ , the Kisin variety is trivial, and so we can associate a shape to a pair $(\bar{\rho}, \tau)$.

There are 25 elements to be analyzed in $\mathrm{Adm}(2, 1, 0)$ (cf. Table 1). Due to an additional symmetry, we are able to reduce our analysis to nine cases. The shorter the length of the shape the more complicated the deformation ring is. In seven cases (length ≥ 2), the deformation ring admits a simple description (see Table 6). The remaining two cases require separate analysis undertaken in §8. Our strategy is as follows:

- (1) Classify all Kisin modules of shape $\tilde{w} \in \mathrm{Adm}(2, 1, 0)$ over $\overline{\mathbb{F}}_p$ (Theorem 2.22);
- (2) For $\overline{\mathfrak{M}}$ of shape \tilde{w} , construct the universal deformation space with height conditions (Theorem 4.14);
- (3) Impose monodromy condition on the universal family (§5).

Steps (1) and (2) generalize techniques of [Bre14, CDMb, EGS15] used to compute tamely Barsotti-Tate deformation rings for GL_2 . Step (2) amounts to constructing local coordinates for the Pappas-Zhu local model for $(\mathrm{GL}_3, \mu = (2, 1, 0), \text{Iwahori level})$ (cf. [CL]) and requires a more systematic approach to the p -adic convergence algorithm employed by [Bre14, CDMb, LM].

Step (3) requires a genuinely new method not present in the tamely Barsotti-Tate case where the link between moduli of finite flat groups schemes and Galois representations is stronger. Kisin [Kis06] characterized when a torsion-free Kisin module \mathfrak{M} over \mathbb{Z}_p comes from a crystalline representation in terms of the poles of a monodromy operator $N_{\mathfrak{M}^{\mathrm{rig}}}$ which is naturally defined on the extension $\mathfrak{M}^{\mathrm{rig}}$ of \mathfrak{M} to the rigid analytic unit ball. This condition on the poles of the monodromy operator is a subtle analogue of Griffiths transversality in p -adic Hodge theory. While one cannot compute $N_{\mathfrak{M}^{\mathrm{rig}}}$ completely, it is possible to give an explicit approximation using the genericity condition on τ . The *error term* turns out to be good enough to understand the geometry of the deformation rings.

Global applications. Using Kisin-Taylor-Wiles patching methods, the local information on the Galois deformation spaces leads to new modularity results and the Serre weight conjectures. To state these results, we fix a global setup (cf. §7.1 and remark that the weight part of Serre's conjecture is expected to be independent of the global setup). Our proofs only use of the existence of *patching functors* in the sense of [EGS15], [GHS] verifying certain axioms (Definition 7.11) and so our results should carry over to many other situations as well.

Let F/\mathbb{Q} be a CM field with totally real subfield F^+ . Assume that p splits completely in F . Let $\bar{\tau} : \mathrm{Gal}(\bar{F}/F) \rightarrow \mathrm{GL}_3(\bar{\mathbb{F}})$ be a continuous irreducible representation. Let G be a unitary group over F^+ which is isomorphic to $U(3)$ at each infinite place and split above p . Attached to this data, there is a well known notion of modularity for $\bar{\tau}$ (cf. Definition 7.1). Roughly speaking, we can find a tame level U^p in the finite adèlic points $G(\mathbb{A}_F^{\infty, p})$ of G and a maximal ideal $\mathfrak{m}_{\bar{\tau}}$ in the anemic Hecke algebra \mathbb{T} acting on the space of mod p algebraic automorphic forms $S(U^p, \mathbb{F})$ with infinite level at p , such that $S(U^p, \mathbb{F})_{\mathfrak{m}_{\bar{\tau}}} \neq 0$.

In order to specify the minimal weight for which $\bar{\tau}$ is modular, it is natural to consider local systems on forms on $U(3)$, attached to irreducible mod p -representations of $G(\mathcal{O}_{F_p^+})$ (also called *Serre weights*). We let $W(\bar{\tau})$ be the set of modular weights (i.e., the set of Serre weights V for which $\mathrm{Hom}_{G(\mathcal{O}_{F_p^+})}(V, S(U^p, \mathbb{F})_{\mathfrak{m}_{\bar{\tau}}}) \neq 0$). We define a condition of genericity for Serre weights (cf. Definition 7.2) and write $W_{\mathrm{gen}}(\bar{\tau})$ for the set of generic modular weights.

If $\bar{\tau}$ is semisimple at each place above p , then there is a set of conjectural weights $W^?(\bar{\tau})$ defined in [Her09, GHS] which only depends on the restriction of $\bar{\tau}$ to the inertia subgroups at the primes above p .

Theorem 1.2. *Let $\bar{\tau} : G_F \rightarrow \mathrm{GL}_3(\mathbb{F})$ be a continuous Galois representation, verifying the Taylor-Wiles conditions (cf. Definition 7.3). Assume that $\bar{\tau}|_{G_{F_v}}$ is semisimple and 6-generic for all $v \mid p$ (cf. Definition 3.6), that $\bar{\tau}$ is automorphic of some generic Serre weight, and that $\bar{\tau}$ has split ramification outside p . Then*

$$W_{\mathrm{gen}}(\bar{\tau}) = W^?(\bar{\tau}).$$

When $\bar{\tau}$ is irreducible at each prime above p , this is proven in [EGH13] using the technique of weight cycling and without any Taylor-Wiles conditions. The inclusion $W_{\mathrm{gen}}(\bar{\tau}) \subset W^?(\bar{\tau})$ (weight elimination) is proven in [EGH13, HLM, MP]. Recent improvements in weight elimination results show that $W_{\mathrm{gen}}(\bar{\tau})$ can be replaced by $W(\bar{\tau})$ and ‘automorphic of some generic Serre weight’ with just ‘automorphic’ in almost all cases, see Remark 7.10.

If $[F^+ : \mathbb{Q}] = d$, there are 9^d conjectural weights appearing in $W^?(\bar{\tau})$, 6^d of which are called *obvious* weights since they are directly related to the Hodge-Tate weights of “obvious” crystalline lifts of $(\bar{\tau}|_{G_{F_v}})_{v|p}$. The precise relation between Serre weights of $\bar{\tau}$ and Hodge-Tate weights of crystalline lifts of $\bar{\rho}$ was first made precise in [Gee11] and the obvious weights are shown to be modular in [BLGG] using global methods (namely, modularity lifting techniques) under the assumption that $\bar{\tau}$ is modular of a Fontaine-Laffaille weight.

The remaining weights in $W^?(\bar{\tau})$ are more mysterious and are referred to as *shadow weights*. The modularity of the shadow weights lies deeper than that of the obvious weights, in part, because modularity of a shadow weight cannot be detected by modularity lifting alone but requires characteristic p information. It is at this point that the computation of the monodromy operator appears to play a critical role. The proof of Theorem 1.2 builds on the Breuil-Mézard philosophy introduced in [GK14]. The patching techniques of Gee-Kisin [GK14], Emerton-Gee [EG14] connect the geometry of the local deformation rings to modularity questions. We use geometric information about the local deformation rings, especially the geometry of their special fibers, to prove the modularity for the shadow weights.

Theorem 1.2 is stated only for $\bar{\tau}$ which are semisimple above p because those are the only representations for which there is an explicit conjecture. Our computations, together with work of [HLM, MP], suggest a set $W^?(\bar{\rho})$ for non-semisimple $\bar{\rho}$ for which the Theorem should hold. We give one example in Proposition 7.17 and will return to this question in future work. We also deduce from Table 8 counterexamples when $\bar{\rho}$ is not semisimple to Conjecture 4.3.2 of [Gee11], which

predicts Serre weights in terms of the existence of crystalline lifts. The conjecture is reformulated in [GHS, Conjecture 5.1.6].

The local information on the deformation rings (Theorem 1.1), namely the connectedness of their generic fiber, lets us deduce new modularity lifting theorems.

Theorem 1.3. *Let $r : G_F \rightarrow \mathrm{GL}_3(\mathcal{O})$ be a Galois representation and write $\bar{r} : G_F \rightarrow \mathrm{GL}_3(\mathbb{F})$ for its associated residual representation.*

Assume that:

- (1) *p splits completely in F^+ ;*
- (2) *r is conjugate self-dual and unramified almost everywhere;*
- (3) *for all places $w \in \Sigma_p$, the representation $r|_{G_{F_w}}$ is potentially crystalline, with parallel Hodge type $(2, 1, 0)$ and with strongly generic tame inertial type $\tau_{\Sigma_p^+} = \otimes_{v \in \Sigma_p^+} \tau_v$ (cf. Definition 2.1);*
- (4) *\bar{r} verifies the Taylor-Wiles conditions (cf. Definition 7.3, in particular \bar{r} is absolutely irreducible) and \bar{r} has split ramification;*
- (5) *$\bar{r} \cong \bar{r}_i(\pi)$ for a RACSDC representation π of $\mathrm{GL}_3(\mathbb{A}_F)$ with trivial infinitesimal character and such that $\otimes_{v \in \Sigma_p^+} \sigma(\tau_v)$ is a K -type for $\otimes_{v \in \Sigma_p^+} \pi_v$.*

Then r is automorphic.

In Theorem 1.3, we do not assume that $\bar{\rho}$ is semisimple nor do we make any potential diagonalizability assumptions. We also allow any tame type not just principal series types. Assumption (1) can be relaxed to the condition that p is unramified in F^+ . This requires new automorphic techniques which will be discussed in a companion paper ([LLHLM]). We believe also that the genericity assumptions on the type τ can be weakened.

Our results shed light on other questions in mod p and p -adic Langlands as well. For example, Breuil [Bre14] formulates a conjecture, based on calculations of tamely Barsotti-Tate deformation rings for GL_2 , on integral lattices in tame types cut out by completed cohomology. When a tame principal series representation π of $G(F_p^+)$ appears in the $\mathfrak{m}_{\bar{r}}$ -part (for a globalization of $\bar{\rho}$) of the completed cohomology of an appropriate Shimura curve, the natural integral structure on the cohomology induces an integral structure on the type associated to π . Breuil conjectures that this lattice only depends on the local p -adic Galois representation attached to π by the hypothetical p -adic local Langlands correspondence. Another related conjecture in [Bre14] is a “multiplicity one” statement for cohomology at Iwahori level. Both conjectures for K/\mathbb{Q}_p unramified were proven by Emerton-Gee-Savitt [EGS15] using the Taylor-Wiles method and geometric Breuil-Mézard realized by explicit presentations of tamely Barsotti-Tate deformation rings.

The first author proved a generalization of Breuil’s conjecture to GL_3 in the setting of [EGH13]. In a companion work ([LLHLM]), we address Breuil’s conjecture for GL_3 and K/\mathbb{Q}_p unramified using the methods developed in this paper. We will also extend the global results of this paper to the case of K/\mathbb{Q}_p unramified. For ease of exposition, we restrict ourselves to the case of GL_3 throughout this paper, although §2-4, some of §5, and §6 could be extended to GL_n without serious difficulty.

Overview of the paper. We start in Section §2.1 with the basic formalism of Kisin modules with tame descent data (notion of eigenbasis, genericity and the basic formulas of semilinear algebra). This is further pursued in §2.2, where we obtain a complete classification of Kisin modules with generic descent data in terms of shapes (cf. Definition 2.17 and Theorem 2.22). Section 3 compares the moduli of Kisin modules with generic tame descent data and Galois deformation spaces. The genericity assumption guarantees the triviality of the Kisin variety (Theorem 3.2) and injectivity on tangent spaces. We conclude the section with the notion of shape and genericity for a mod p Galois representation (cf. Definitions 3.3, 3.6) together with a Galois cohomology argument which shows that, under a mild assumption on $\overline{\rho}$ the restriction on tangent spaces to the Kummer extension is fully faithful.

Section 4 and 5 are the technical heart of the paper. In §4, we develop an algorithm to construct a “universal family” of Kisin modules of finite height, lifting a residual Kisin module of a given shape (Theorem 4.1). The strategy is a wide generalization of the methods already appearing in [Bre14, CDMa, LM]. An algorithm is described to construct a *gauge basis* on the universal family in §4.1 on which we then impose the p -adic Hodge type conditions (cf. Table 4 and Theorem 4.14).

In §5, we endow the rigid analytification of the universal family of Kisin modules with a canonical monodromy operator, which we determine up to an error term which is divisible by a power of p (Theorem 5.6); by imposing the monodromy to have logarithmic poles (Proposition 5.3), we finally obtain explicit equations for the moduli of Kisin modules with monodromy (Table 5), hence for the Galois deformation ring (§5.2, Corollary 5.14 and Table 6).

Section 6 extends the results of §5 to other tame types (cf. Theorem 6.9). We generalize the formalism of “base change” for deformation rings as developed in [EGS15] in dimension 2.

The main global applications are discussed in §7. Via Kisin-Taylor-Wiles patching and the formalism of patched functors (as introduced in [EGS15]), we prove the main theorems discussed above. Section 8 is devoted to the analysis of the monodromy condition when the shape has length ≤ 1 where the computation becomes more involved. In the Appendix, we collect tables summarizing our results.

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1.1. Notation. If F is any field, we write $G_F \stackrel{\text{def}}{=} \text{Gal}(\overline{F}/F)$ for the absolute Galois group, where \overline{F} is a separable closure of F . If F is moreover a p -adic field, we write I_F to denote the inertia subgroup of G_F .

We fix once and for all an algebraic closure $\overline{\mathbb{Q}}$ of \mathbb{Q} . All number fields are considered as subfield of our fixed $\overline{\mathbb{Q}}$. Similarly, if $\ell \in \mathbb{Q}$ is a prime, we fix algebraic closures $\overline{\mathbb{Q}}_\ell$ as well as embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$. All finite extensions of \mathbb{Q}_ℓ will this be considered as subfield in $\overline{\mathbb{Q}}_\ell$. Moreover, the residue field of $\overline{\mathbb{Q}}_\ell$ is denoted by $\overline{\mathbb{F}}_\ell$.

Let $p > 3$ be a prime. For $f > 0$, we let K be the unramified extension of \mathbb{Q}_p of degree f . We write k for its residue field and let $W = W(k)$. We set $e \stackrel{\text{def}}{=} p^f - 1$ and consider the Eisenstein polynomials $E(u) \stackrel{\text{def}}{=} u^e + p \in K[u]$ and $P(v) \stackrel{\text{def}}{=} v + p \in K[v]$ where $v = u^e$. We fix a root $\pi \stackrel{\text{def}}{=} (-p)^{\frac{1}{e}} \in \overline{K}$, define the extension $L = K(\pi)$ and set $\Delta \stackrel{\text{def}}{=} \text{Gal}(L/K)$. The choice of the root π let us define a character

$$\begin{aligned} \tilde{\omega}_\pi : \Delta &\rightarrow W^\times \\ g &\mapsto \frac{g(\pi)}{\pi} \end{aligned}$$

whose associated residual character is denoted by ω_π . In particular, for $f = 1$, ω_π is the mod p cyclotomic character, which will be simply denoted by ω . The p -adic cyclotomic character will be denoted by $\varepsilon : G_{\mathbb{Q}_p} \rightarrow \mathbb{Z}_p^\times$. If F_w/\mathbb{Q}_p is a finite extension and $W_{F_w} \leq G_{F_w}$ denotes the Weil group we normalize Artin's reciprocity map $\text{Art}_{F_w} : F_w^\times \rightarrow W_{F_w}^{\text{ab}}$ in such a way that the geometric Frobenius elements are sent to uniformizers.

Let E be a finite extension of \mathbb{Q}_p . We write \mathcal{O} for its ring of integers, fix an uniformizer $\varpi \in \mathcal{O}$ and let $\mathfrak{m}_E = (\varpi)$. We write $\mathbb{F} \stackrel{\text{def}}{=} \mathcal{O}/\mathfrak{m}_E$ for its maximal ideal. We will always assume that E is sufficiently large, i.e. that any embedding $\sigma : K \hookrightarrow \overline{\mathbb{Q}}_p$ factors through $E \subset \overline{\mathbb{Q}}_p$. In particular, any embedding $\sigma : k \hookrightarrow \overline{\mathbb{F}}_p$ factors through \mathbb{F} .

Let $\rho : G_K \rightarrow \text{GL}_n(E)$ be a p -adic, de Rham Galois representation. For $\sigma : K \hookrightarrow \overline{\mathbb{Q}}_p$, we define $\text{HT}_\sigma(\rho)$ to be the multiset of σ -labeled Hodge-Tate weights of ρ , i.e. the set of integers $-i$ such that $\dim_E(\rho \otimes_{\sigma, K} \mathbb{C}_p(i))^{G_K} \neq 0$ (with the usual notation for Tate twists). In particular, we have $\text{HT}_\sigma(\varepsilon) = \{1\}$. We define the *Hodge type* of ρ to be the multiset $(\text{HT}_\sigma(\rho))_{\sigma \in S_K} \in (\mathbb{Z}^n)^{S_K}$ where $S_K \stackrel{\text{def}}{=} \{\sigma \mid \sigma : K \hookrightarrow \overline{\mathbb{Q}}_p\}$ and the *inertial type* of ρ as the isomorphism class of $\text{WD}(\rho)|_{I_K}$, where $\text{WD}(\rho)$ is the Weil-Deligne representation attached to ρ as in [CDT99], Appendix B.1 (in particular,

$\rho \mapsto \mathrm{WD}(\rho)$ is *covariant*). Recall that an inertial type is a morphism $\tau : I_K \rightarrow \mathrm{GL}_n(\mathcal{O})$ with open kernel and which extends to the Weil group W_K of G_K .

We now fix an embedding $\sigma_0 : K \subset E$. The embedding σ_0 induces maps $W \hookrightarrow \mathcal{O}$ and $k \hookrightarrow \mathbb{F}$; we will abuse notation and denote these all by σ_0 . We let φ denote the p -th power Frobenius on k and set $\sigma_j \stackrel{\mathrm{def}}{=} \sigma_0 \circ \varphi^{-j}$. The choice of embedding gives a fundamental character $\omega_f := \sigma_0 \circ \tilde{\omega}_\pi : I_K \rightarrow \mathcal{O}^\times$ of niveau f .

Let S_3 denote the symmetric group on $\{1, 2, 3\}$. We fix an injection $S_3 \hookrightarrow \mathrm{GL}_3(\mathbb{Z})$ by setting $s \mapsto M_s$, where M_s is defined by $(M_s)_{k,m} \stackrel{\mathrm{def}}{=} \delta_{k,s(m)}$ and $\delta_{k,s(m)} \in \{0, 1\}$ is the Kronecker δ specialized at $\{k, s(m)\}$.

2. KISIN MODULES MODULO p

2.1. Kisin modules with descent datum. Let $\tau = \eta_1 \oplus \eta_2 \oplus \eta_3$ be an \mathcal{O}^\times -valued inertial type consisting of pairwise distinct non-trivial characters.

Let $\mathbf{a}_1 = (a_{1,j})_j$, $\mathbf{a}_2 = (a_{2,j})_j$, and $\mathbf{a}_3 = (a_{3,j})_j$ where $0 \leq j \leq f-1$ and $0 \leq a_{k,j} \leq p-1$. For any $0 \leq j \leq f-1$, define

$$\mathbf{a}_k^{(j)} = \sum_{i=0}^{f-1} a_{k,-j+i} p^i.$$

For $1 \leq k \leq 3$, we can write

$$\eta_k = (\omega_f)^{-\mathbf{a}_k^{(0)}}$$

for a unique choice of \mathbf{a}_k . We say that $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ is *associated to* τ . We will need the following genericity assumption throughout the paper.

Definition 2.1. We say that the triple $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ is *generic* if

$$3 \leq |a_{1,j} - a_{2,j}|, |a_{2,j} - a_{3,j}|, |a_{1,j} - a_{3,j}| \leq p-4$$

for all j . We say that an inertial type $\tau = \eta_1 \oplus \eta_2 \oplus \eta_3$ is *generic* if the associated triple $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ is generic. Similarly, we say τ is *strongly generic* if the inequalities (2.1) hold with the bounds 3 and $p-4$ replaced by 5 and $p-6$ respectively.

Let R be an \mathcal{O} -algebra. Any $W \otimes_{\mathbb{Z}_p} R$ -module M decomposes as direct sum of R -modules $M = \bigoplus_{j=0}^{f-1} M^{(j)}$ where $M^{(j)}$ is the submodule such that $(x \otimes 1)m = (1 \otimes \sigma_j(x))m$ for all $m \in M^{(j)}$ and $x \in W$.

For any $g \in \Delta$ and any \mathcal{O} -algebra R , we let \widehat{g} be the automorphism of $(W \otimes_{\mathbb{Z}_p} R)[[u]]$ given by $u \mapsto (\omega_\pi(g) \otimes 1)u$.

Definition 2.2. Let \mathfrak{M}_R be an $(W \otimes_{\mathbb{Z}_p} R)[[u]]$ -module. A *semilinear action* of Δ on \mathfrak{M}_R is collection of \widehat{g} -semilinear bijections $\widehat{g} : \mathfrak{M}_R \rightarrow \mathfrak{M}_R$ for each $g \in \Delta$ such that

$$\widehat{g} \circ \widehat{h} = \widehat{gh}$$

for all $g, h \in \Delta$.

Recall that for any \mathcal{O} -algebra R , we define the Frobenius $\varphi : (W \otimes_{\mathbb{Z}_p} R)[[u]] \rightarrow (W \otimes_{\mathbb{Z}_p} R)[[u]]$ to be trivial on R , the Frobenius on W , and with $\varphi(u) = u^p$.

Definition 2.3. Let R be any \mathcal{O} -algebra. A *Kisin module* with height in $[0, h]$ over R is a finitely generated projective $(W \otimes R)[[u]]$ -module \mathfrak{M}_R together with Frobenius $\phi_{\mathfrak{M}_R} : \varphi^*(\mathfrak{M}_R) \rightarrow \mathfrak{M}_R$ such that the cokernel is killed by $E(u)^h$.

Definition 2.4. A *Kisin module with descent datum* over R is a Kisin module $(\mathfrak{M}_R, \phi_{\mathfrak{M}_R})$ together with a semilinear action of Δ given by $\{\widehat{g}\}_{g \in \Delta}$ which commutes with $\phi_{\mathfrak{M}_R}$, i.e., for all $g \in \Delta$,

$$\varphi^*(\widehat{g}) \circ \phi_{\mathfrak{M}_R} = \phi_{\mathfrak{M}_R} \circ \widehat{g}.$$

Let $\mathfrak{M}_R \cong \bigoplus_{j=0}^{f-1} \mathfrak{M}_R^{(j)}$. We say that the descent datum is of *type* τ if the linear representation of Δ on the R -module satisfies $\mathfrak{M}_R^{(j)}/u\mathfrak{M}_R^{(j)} \cong \tau \otimes_{\mathcal{O}} R$ for each $0 \leq j \leq f-1$.

For any \mathcal{O} -algebra R , let $Y^{[0,h],\tau}(R)$ be the category of Kisin modules over R with height in $[0, h]$, rank 3, and descent datum of type τ . For a given $N \in \mathbb{N}$, it is shown in [CL] that $Y^{[0,h],\tau} \bmod (\mathfrak{m}_E)^N$ is represented by an Artin stack of finite type over $\mathcal{O}/(\mathfrak{m}_E)^N$. The aim of this section is to classify the $\overline{\mathbb{F}}$ -points of $Y^{[0,2],\tau}$ which are reductions of Kisin modules with “Hodge-Tate” weights $(2, 1, 0)$.

Definition 2.5. Let $v \stackrel{\text{def}}{=} u^e$ and $\mathfrak{M}_R \in Y^{[0,2],\tau}(R)$. For $k \in \{1, 2, 3\}$ define $\mathfrak{M}_{R,k}$ to be $(W \otimes R)[[v]]$ submodule of \mathfrak{M}_R on which Δ acts by η_k , i.e., $\mathfrak{M}_{R,k} \stackrel{\text{def}}{=} (\mathfrak{M}_R)^{\Delta=\eta_k}$. Similarly, we define ${}^\varphi\mathfrak{M}_{R,k}$ to be the $(W \otimes R)[[v]]$ submodule of $\varphi^*(\mathfrak{M}_R)$ on which Δ acts by η_k , i.e., ${}^\varphi\mathfrak{M}_{R,k} \stackrel{\text{def}}{=} (\varphi^*(\mathfrak{M}_R))^{\Delta=\eta_k}$.

By considering the decomposition $\mathfrak{M}_R \cong \bigoplus_{j=0}^{f-1} \mathfrak{M}_R^{(j)}$, we write $\mathfrak{M}_{R,k}^{(j+1)}$ for the $R[[v]]$ -submodules of $\mathfrak{M}_R^{(j+1)}$ on which Δ acts by η_k and we write ${}^\varphi\mathfrak{M}_{R,k}^{(j)}$ for the $R[[v]]$ -submodules of $(\varphi^*(\mathfrak{M}_R))^{(j+1)} = \varphi^*(\mathfrak{M}_R^{(j)})$ on which Δ acts by η_k (with the usual convention that $j+1 \stackrel{\text{def}}{=} 0$ if $j = f-1$).

While we have made a choice of global ordering η_1, η_2, η_3 , it will be important for uniform statements to order things (possibly) differently at each embedding $\sigma_j : K \rightarrow E$. We introduce this local ordering now.

Definition 2.6. Let $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ be a triple as in Definition 2.1. The *orientation* of $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ is the f -tuple $(s_j)_j \in S_3^f$ such that

$$\mathbf{a}_{s_j(1)}^{(j)} > \mathbf{a}_{s_j(2)}^{(j)} > \mathbf{a}_{s_j(3)}^{(j)}.$$

If τ is an inertial type as above, the *orientation* of τ is defined to be the orientation of the triple $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ associated to it. In this case, we say that s_j is the *orientation at* j of τ .

Under the genericity condition (2.1), the orientation at j is determined by the values of $a_{1,f-j-1}$, $a_{2,f-j-1}$, $a_{3,f-j-1}$ which are the dominant terms of $\mathbf{a}_1^{(j)}$, $\mathbf{a}_2^{(j)}$, $\mathbf{a}_3^{(j)}$ respectively. In particular, we have

$$(2.1) \quad a_{s_j(1),f-j-1} > a_{s_j(2),f-j-1} > a_{s_j(3),f-j-1}.$$

For any $\mathfrak{M} \in Y^{[0,h],\tau}(R)$, we have the following commutative diagram relating the different isotypic components:

(2.2)

$$\begin{array}{ccccccc}
 \varphi\mathfrak{M}_{s_j(3)}^{(j-1)} & \xrightarrow{u^{e-(\mathbf{a}_{s_j(1)}^{(j)}-\mathbf{a}_{s_j(3)}^{(j)})}} & \varphi\mathfrak{M}_{s_j(1)}^{(j-1)} & \xrightarrow{u^{\mathbf{a}_{s_j(1)}^{(j)}-\mathbf{a}_{s_j(2)}^{(j)}}} & \varphi\mathfrak{M}_{s_j(2)}^{(j-1)} & \xrightarrow{u^{\mathbf{a}_{s_j(2)}^{(j)}-\mathbf{a}_{s_j(3)}^{(j)}}} & \varphi\mathfrak{M}_{s_j(3)}^{(j-1)} \\
 \downarrow \phi_{\mathfrak{M},s_j(3)}^{(j-1)} & & \downarrow \phi_{\mathfrak{M},s_j(1)}^{(j-1)} & & \downarrow \phi_{\mathfrak{M},s_j(2)}^{(j-1)} & & \downarrow \phi_{\mathfrak{M},s_j(3)}^{(j-1)} \\
 \mathfrak{M}_{s_j(3)}^{(j)} & \xrightarrow{u^{e-(\mathbf{a}_{s_j(1)}^{(j)}-\mathbf{a}_{s_j(3)}^{(j)})}} & \mathfrak{M}_{s_j(1)}^{(j)} & \xrightarrow{u^{\mathbf{a}_{s_j(1)}^{(j)}-\mathbf{a}_{s_j(2)}^{(j)}}} & \mathfrak{M}_{s_j(2)}^{(j)} & \xrightarrow{u^{\mathbf{a}_{s_j(2)}^{(j)}-\mathbf{a}_{s_j(3)}^{(j)}}} & \mathfrak{M}_{s_j(3)}^{(j)}
 \end{array}$$

where the composition along each row is multiplication by u^e and the vertical arrows are induced by $\phi_{\mathfrak{M}}$. All the maps in the diagram are injective (again, with the standard convention that $j-1 \stackrel{\text{def}}{=} f-1$ if $j=0$). In particular, any one of the three maps $\phi_{\mathfrak{M},1}^{(j-1)}, \phi_{\mathfrak{M},2}^{(j-1)}, \phi_{\mathfrak{M},3}^{(j-1)}$ determines the other two. We choose to focus on $\phi_{\mathfrak{M},s_j(3)}^{(j-1)}$. We discuss in more detail at the end of the section how the Frobenii $\phi_{\mathfrak{M},s_j(k)}^{(j-1)}$, for $1 \leq k \leq 3$, are related (Proposition 2.24).

Remark 2.7. The submodule $\varphi\mathfrak{M}_k$ of $\varphi^*(\mathfrak{M})$ is NOT the same as the Frobenius pullback of \mathfrak{M}_k . In particular, $\phi_{\mathfrak{M},k}$ does not define a semilinear endomorphism of \mathfrak{M}_k . It is merely a linear map from $\varphi\mathfrak{M}_k \rightarrow \mathfrak{M}_k$. This fact is reflected again in the change of basis formula (Proposition 2.15).

We want to consider $(\phi_{\mathfrak{M},s_{j+1}(3)}^{(j)})_j$ as a collection of matrices with respect to a choice of bases. We will refine the basis further in the next section.

Definition 2.8. Let $\mathfrak{M} \in Y^{[0,2],\tau}(R)$. An *eigenbasis* $\beta := \{\beta^{(j)}\}_j$ for \mathfrak{M} is a collection of bases $\beta^{(j)} = (f_1^{(j)}, f_2^{(j)}, f_3^{(j)})$ of each $\mathfrak{M}^{(j)}$ such that $f_k^{(j)} \in \mathfrak{M}_k^{(j)}$ for each $k \in \{1, 2, 3\}$.

Lemma 2.9. If $\beta = \left\{ \left(f_1^{(j)}, f_2^{(j)}, f_3^{(j)} \right) \right\}_j$ is an eigenbasis for \mathfrak{M} , then for any $0 \leq j \leq f-1$,

$$\beta_{s_j(3)}^{(j)} := \left(u^{\mathbf{a}_{s_j(1)}^{(j)}-\mathbf{a}_{s_j(3)}^{(j)}} f_{s_j(1)}^{(j)}, u^{\mathbf{a}_{s_j(2)}^{(j)}-\mathbf{a}_{s_j(3)}^{(j)}} f_{s_j(2)}^{(j+1)}, f_{s_j(3)}^{(j)} \right),$$

is a basis for $\mathfrak{M}_{s_j(3)}^{(j)}$. Similarly,

$$\varphi\beta_{s_j(3)}^{(j-1)} := \left(u^{\mathbf{a}_{s_j(1)}^{(j)}-\mathbf{a}_{s_j(3)}^{(j)}} \otimes f_{s_j(1)}^{(j-1)}, u^{\mathbf{a}_{s_j(2)}^{(j)}-\mathbf{a}_{s_j(3)}^{(j)}} \otimes f_{s_{j-1}(2)}^{(j-1)}, 1 \otimes f_{s_j(3)}^{(j-1)} \right)$$

is a basis for $\varphi\mathfrak{M}_{s_{j+1}(3)}^{(j-1)}$.

Remark 2.10. We always order an eigenbasis $\beta^{(j)}$ with respect to the ordering on the characters η_1, η_2, η_3 . On the other hand, when we work with the isotypic pieces $\mathfrak{M}_{s_j(3)}^{(j)}$ we order our bases using the orientation (s_j) of τ . It will be important to keep track of this difference.

Definition 2.11. Given an eigenbasis β for \mathfrak{M} , the matrix $C^{(j)}$ of $\phi_{\mathfrak{M}}^{(j)}$ with respect to $\beta^{(j)}$ is defined to be the matrix of

$$\phi_{\mathfrak{M}}^{(j)} : \varphi^*(\mathfrak{M}^{(j)}) \rightarrow \mathfrak{M}^{(j+1)}$$

with respect to the bases $\varphi^*(\beta^{(j)})$ and $\beta^{(j+1)}$.

The matrix $A^{(j)}$ of $\phi_{\mathfrak{M}, s_{j+1}(3)}^{(j)}$ with respect to $\beta^{(j)}$ is defined to be the matrix of

$$\phi_{\mathfrak{M}, s_{j+1}(3)}^{(j)} : \varphi \mathfrak{M}_{s_{j+1}(3)}^{(j)} \rightarrow \mathfrak{M}_{s_{j+1}(3)}^{(j+1)}$$

with respect the bases $\varphi \beta_{s_{j+1}(3)}^{(j)}$ and $\beta_{s_{j+1}(3)}^{(j+1)}$ from Lemma 2.9.

It is customary to write $C^{(j)} = \text{Mat}_{\beta}(\phi_{\mathfrak{M}}^{(j)})$ and $A^{(j)} = \text{Mat}_{\beta}(\phi_{\mathfrak{M}, s_{j+1}(3)}^{(j)})$ for short. To understand how $C^{(j)}$ and $A^{(j)}$ relate to each other we define the following conjugation action by diagonal matrices:

Definition 2.12. For any $b_1, b_2, b_3 \in \mathbb{Z}$ and for any $M \in \text{Mat}_3(R((u)))$, we define

$$\text{Ad}(u^{b_1}, u^{b_2}, u^{b_3})(M) := \begin{pmatrix} u^{b_1} & 0 & 0 \\ 0 & u^{b_2} & 0 \\ 0 & 0 & u^{b_3} \end{pmatrix} M \begin{pmatrix} u^{-b_1} & 0 & 0 \\ 0 & u^{-b_2} & 0 \\ 0 & 0 & u^{-b_3} \end{pmatrix}.$$

For any $0 \leq j \leq f-1$ and any $M \in \text{Mat}_3(R((u)))$, we define conjugation with orientation by

$$\text{Ad}_{s_j}(u^{\mathbf{a}^1}, u^{\mathbf{a}^2}, u^{\mathbf{a}^3})(M) := s_j \left(\text{Ad} \left(u^{\mathbf{a}_{s_j(1)}^{(j)}}, u^{\mathbf{a}_{s_j(2)}^{(j)}}, u^{\mathbf{a}_{s_j(3)}^{(j)}} \right) (M) \right) s_j^{-1}$$

and

$$\text{Ad}_{s_j}^{-1}(u^{\mathbf{a}^1}, u^{\mathbf{a}^2}, u^{\mathbf{a}^3})(M) := \text{Ad} \left(u^{-\mathbf{a}_{s_j(1)}^{(j)}}, u^{-\mathbf{a}_{s_j(2)}^{(j)}}, u^{-\mathbf{a}_{s_j(3)}^{(j)}} \right) (s_j^{-1} M s_j).$$

The above conjugation corresponds to “removing the descent datum.” For $\mathfrak{M} \in Y^{[0, h], \tau}(R)$, the following Proposition relates the matrix for $\phi_{\mathfrak{M}}^{(j)}$ to the matrix for $\phi_{\mathfrak{M}, s_{j+1}(3)}^{(j)}$.

Proposition 2.13. Let β be an eigenbasis for \mathfrak{M} and (s_j) be the orientation of τ . Let $A^{(j)} = \text{Mat}_{\beta}(\phi_{\mathfrak{M}, s_{j+1}(3)}^{(j)})$ be as in Definition 2.11. Then

$$(2.3) \quad C^{(j)} = \text{Mat}_{\beta}(\phi_{\mathfrak{M}}^{(j)}) = \text{Ad}_{s_{j+1}}(u^{\mathbf{a}^1}, u^{\mathbf{a}^2}, u^{\mathbf{a}^3})(A^{(j)})$$

Proof. This is straightforward from Definition 2.11 and Lemma 2.9. \square

For any \mathcal{O} -algebra R , define

- $L \text{GL}_3(R) := \text{GL}_3(R((v)))$
- $L^+ \text{GL}_3(R) := \text{GL}_3(R[[v]])$
- $\mathcal{I}(R) := \{M \in L^+ \text{GL}_3(R) \mid M \bmod v \text{ is upper triangular} \}$

- $\mathcal{I}_1(R) := \{M \in L^+ \mathrm{GL}_3(R) \mid M \bmod v \equiv \mathrm{Id}\}$
- $\mathcal{D}_3(R) := \{M \in L^+ \mathrm{GL}_3(R) \mid M \bmod v^3 \text{ is diagonal}\}.$

Lemma 2.14. *Let $I \in \mathrm{GL}_3(R(\!(v)\!))$. For any integers b_1, b_2, b_3 with $e > b_1 - b_3 > b_2 - b_3 > 0$, consider*

$$D = \mathrm{Ad}(u^{b_1}, u^{b_2}, u^{b_3})(I).$$

Then $D \in \mathrm{GL}_3(R(\![u]\!))$ if and only if $I \in \mathcal{I}(R)$.

Proof. The proof is a straightforward computation. \square

We will now describe the effect of change of basis for the eigenbasis coordinates. Recall that $(s_j) \in S_3^f$ is the orientation of τ (2.6), and we associate to s_j the corresponding permutation matrix in $\mathrm{GL}_3(\mathcal{O})$ as described in §1.1.

Proposition 2.15. *Let R be an \mathcal{O} -algebra. Let $\mathfrak{M} \in Y^{[0,2],\tau}(R)$ together with two eigenbases $\beta_1^{(j)} := (f_1^{(j)}, f_2^{(j)}, f_3^{(j)})$ and $\beta_2^{(j)} := (f_1'^{(j)}, f_2'^{(j)}, f_3'^{(j)})$ related by*

$$(f_1^{(j)}, f_2^{(j)}, f_3^{(j)}) D^{(j)} = (f_1'^{(j)}, f_2'^{(j)}, f_3'^{(j)})$$

with $D^{(j)} \in \mathrm{GL}_3(R(\![u]\!))$. Let us write $A_1^{(j)} \stackrel{\mathrm{def}}{=} \mathrm{Mat}_{\beta_1}(\phi_{\mathfrak{M}, s_{j+1}(3)}^{(j)})$ and $A_2^{(j)} \stackrel{\mathrm{def}}{=} \mathrm{Mat}_{\beta_2}(\phi_{\mathfrak{M}, s_{j+1}(3)}^{(j)})$ as in Definition 2.11. Then

$$(2.4) \quad A_2^{(j)} = I^{(j+1)} A_1^{(j)} (s_{j+1}^{-1} s_j (I^{(j), \varphi}) s_j^{-1} s_{j+1})$$

where, for all $0 \leq j \leq f-1$, we have $I^{(j)} \stackrel{\mathrm{def}}{=} \mathrm{Ad}_{s_j}^{-1}(u^{\mathbf{a}_1}, u^{\mathbf{a}_2}, u^{\mathbf{a}_3})(D^{(j)}) \in \mathcal{I}(R)$ and

$$I^{(j), \varphi} \stackrel{\mathrm{def}}{=} \mathrm{Ad}(v^{a_{s_j(1), f-j-1}}, v^{a_{s_j(2), f-j-1}}, v^{a_{s_j(3), f-j-1}})(\varphi(I^{(j)})^{-1}).$$

Proof. The proof is a direct computation using Proposition 2.13. More precisely, let us write $C_i^{(j)} \stackrel{\mathrm{def}}{=} \mathrm{Mat}_{\beta_i}(\phi_{\mathfrak{M}}^{(j)})$ for $i \in \{1, 2\}$ as in Definition 2.11. We have

$$(2.5) \quad C_2^{(j)} = D^{(j+1)} C_1^{(j)} \varphi(D^{(j)})^{-1}.$$

Since $D^{(j)}$ respects the descent datum, $I^{(j)} := \mathrm{Ad}_{s_j}^{-1}(u^{\mathbf{a}_1}, u^{\mathbf{a}_2}, u^{\mathbf{a}_3})(D^{(j)})$ is in $\mathcal{I}(R)$. Using (2.5) and Proposition 2.13, one obtains:

$$\begin{aligned} \mathrm{Ad}_{s_{j+1}}(u^{\mathbf{a}_1}, u^{\mathbf{a}_2}, u^{\mathbf{a}_3})(A_2^{(j)}) &= D^{(j+1)} C_1^{(j)} \varphi(D^{(j)})^{-1} \\ &= D^{(j+1)} \left(\mathrm{Ad}_{s_{j+1}}(u^{\mathbf{a}_1}, u^{\mathbf{a}_2}, u^{\mathbf{a}_3})(A_1^{(j)}) \right) \varphi(D^{(j)})^{-1} \end{aligned}$$

Conjugating on both sides, we further deduce that

$$(2.6) \quad A_2^{(j)} = I^{(j+1)} A_1^{(j)} \mathrm{Ad}_{s_{j+1}}^{-1}(u^{\mathbf{a}_1}, u^{\mathbf{a}_2}, u^{\mathbf{a}_3})(\varphi(D^{(j)})^{-1}).$$

We now study the final term. Let $s_{j+1,j} := s_{j+1}^{-1} s_j$. We have

$$\begin{aligned}
 (2.7) \quad & \text{Ad}_{s_{j+1}}^{-1}(u^{\mathbf{a}_1}, u^{\mathbf{a}_2}, u^{\mathbf{a}_3}) \left(\varphi \left(D^{(j)} \right)^{-1} \right) \\
 &= \text{Ad} \left(u^{-\mathbf{a}_{s_{j+1}^{(1)}}^{(j+1)}}, u^{-\mathbf{a}_{s_{j+1}^{(2)}}^{(j+1)}}, u^{-\mathbf{a}_{s_{j+1}^{(3)}}^{(j+1)}} \right) \left(s_{j+1,j} \text{Ad} \left(u^{p\mathbf{a}_{s_j^{(1)}}^{(j)}}, u^{p\mathbf{a}_{s_j^{(2)}}^{(j)}}, u^{p\mathbf{a}_{s_j^{(3)}}^{(j)}} \right) (\varphi(I^{(j)})^{-1}) s_{j+1,j}^{-1} \right) \\
 &= s_{j+1,j} \text{Ad} \left(v^{a_{s_j^{(1)},f-j-1}}, v^{a_{s_j^{(2)},f-j-1}}, v^{a_{s_j^{(3)},f-j-1}} \right) (\varphi(I^{(j)})^{-1}) s_{j+1,j}^{-1}.
 \end{aligned}$$

where the last step follows from $p\mathbf{a}_k^{(j)} - \mathbf{a}_k^{(j+1)} = a_{k,f-j-1}$ and

$$s_{j+1,j}^{-1} \begin{pmatrix} u^{\mathbf{a}_{s_{j+1}^{(1)}}^{(j+1)}} & 0 & 0 \\ 0 & u^{\mathbf{a}_{s_{j+1}^{(2)}}^{(j+1)}} & 0 \\ 0 & 0 & u^{\mathbf{a}_{s_{j+1}^{(3)}}^{(j+1)}} \end{pmatrix} s_{j+1,j} = \begin{pmatrix} u^{\mathbf{a}_{s_j^{(1)}}^{(j+1)}} & 0 & 0 \\ 0 & u^{\mathbf{a}_{s_j^{(2)}}^{(j+1)}} & 0 \\ 0 & 0 & u^{\mathbf{a}_{s_j^{(3)}}^{(j+1)}} \end{pmatrix}.$$

The conclusion follows by combining (2.6) and (2.7). \square

Proposition 2.16. *Assume that τ is generic (2.1). Let $I^{(j)} \in \mathcal{I}(R)$ be as in Proposition 2.15. Then*

$$I^{(j),\varphi} = \text{Ad} \left(v^{a_{s_j^{(1)},f-j-1}}, v^{a_{s_j^{(2)},f-j-1}}, v^{a_{s_j^{(3)},f-j-1}} \right) (\varphi(I^{(j)})^{-1}) \in \mathcal{D}_3(R).$$

Proof. By the genericity assumptions and choice of orientation, $a_{s_j^{(1)},f-j-1} - a_{s_j^{(2)},f-j-1} \geq 3$ and

$$p - 4 \geq a_{s_j^{(1)},f-j-1} - a_{s_j^{(3)},f-j-1} > a_{s_j^{(2)},f-j-1} - a_{s_j^{(3)},f-j-1} \geq 3.$$

Since $(I^{(j)})^{-1} \in \mathcal{I}(R)$, the entries of $\varphi(I^{(j)})^{-1}$ below the diagonal are divisible by v^p . A direct computation then shows that

$$\text{Ad} \left(v^{a_{s_j^{(1)},f-j-1}}, v^{a_{s_j^{(2)},f-j-1}}, v^{a_{s_j^{(3)},f-j-1}} \right) (\varphi(I^{(j)})^{-1}) \in \mathcal{D}_3(R).$$

\square

2.2. Classification over \mathbb{F} . We keep the notations of the previous section except we now work over \mathbb{F} as opposed to \mathcal{O} . We are now ready to define the *shape* (or *genre* in French) of a Kisin module. Let T be the diagonal torus of GL_3 and let $N_{\text{GL}_3}(T)$ denote the normalizer of T . The (extended) affine Weyl group of GL_3 is given by

$$\widetilde{W} := N_{\text{GL}_3}(T)(\mathbb{F}((v)))/T(\mathbb{F}[[v]]).$$

Recall that \widetilde{W} sits in an exact sequence

$$0 \rightarrow X_*(T) \rightarrow \widetilde{W} \rightarrow S_3 \rightarrow 0$$

where S_3 is the ordinary Weyl group of GL_3 and $X_*(T) \cong \mathbb{Z}^3$ are the cocharacters of T . For any finite extension \mathbb{F}' of \mathbb{F} , Bruhat-Tits theory gives the following double coset decomposition

$$(2.8) \quad L\mathrm{GL}_3(\mathbb{F}') = \bigcup_{\tilde{w} \in \widetilde{W}} \mathcal{I}(\mathbb{F}') \tilde{w} \mathcal{I}(\mathbb{F}').$$

Definition 2.17. Let $\mathbf{w} = (\tilde{w}_0, \tilde{w}_1, \dots, \tilde{w}_{f-1}) \in \widetilde{W}^f$. A Kisin module $\mathfrak{M} \in Y^{[0,h],\tau}(\mathbb{F}')$ has *shape* \mathbf{w} if for any eigenbasis β , the matrices $(A^{(j)})_j = (\mathrm{Mat}_\beta(\phi_{\mathfrak{M},s_{j+1}(3)}^{(j)}))_j$ have the property that $A^{(j)} \in \mathcal{I}(\mathbb{F}') \tilde{w}_j \mathcal{I}(\mathbb{F}')$.

Independence from the choice of basis is clear from Propositions 2.15 and 2.16. The motivation for this definition comes from the theory of local models and a corresponding stratification there. We now give an overview of this connection which is described in detail in joint work of the third author [CL]. We turn our attention to the study of Kisin modules with “parallel” weight $(2, 1, 0)$. Precisely, let $\mu = (\mu_j)$ with $\mu_j = (2, 1, 0)$ for all j considered as a geometric cocharacter of $\mathrm{Res}_{K/\mathbb{Q}_p} \mathrm{GL}_3$. Then, [CL] constructs a closed substack $Y^{\mu,\tau} \subset Y^{[0,2],\tau}$ together with a “local model diagram”

$$\begin{array}{ccc} & \tilde{Y}^{\mu,\tau} & \\ \pi \swarrow & & \searrow \Psi \\ Y^{\mu,\tau} & & M(\mu), \end{array}$$

where $M(\mu)$ is the Pappas-Zhu local model for $\mathrm{Res}_{K/\mathbb{Q}_p} \mathrm{GL}_3$ with Iwahori level structure and cocharacter μ . Both π and Ψ are smooth maps. For any $\mathbf{w} \in \widetilde{W}^f$, define $\overline{Y}_{\mathbf{w}}^{\mu,\tau}(\overline{\mathbb{F}}) \subset \overline{Y}^{\mu,\tau}(\overline{\mathbb{F}})$ to be the set of points with shape \mathbf{w} .

Proposition 2.18. Let $\overline{M}(\mu)$ and $\overline{Y}^{\mu,\tau}$ denote the special fibers of $M(\mu)$ and $Y^{\mu,\tau}$. Then $\overline{M}(\mu)$ has a stratification by locally closed affine Schubert varieties $S_{\mathbf{w}}^0$ indexed by elements $\mathbf{w} \in \widetilde{W}^f$. Furthermore, the set of points $\overline{Y}_{\mathbf{w}}^{\mu,\tau}(\overline{\mathbb{F}}) \subset \overline{Y}^{\mu,\tau}(\overline{\mathbb{F}})$ of shape \mathbf{w} is given by $\pi(\Psi^{-1}(S_{\mathbf{w}}^0))$.

The set of $\mathbf{w} \in \widetilde{W}^f$ such that $S_{\mathbf{w}}^0$ is a subscheme of $\overline{M}(\mu)$ is given by the μ -admissible set $\mathrm{Adm}(\mu) = \prod_j \mathrm{Adm}(2, 1, 0)$ (see [CL, §5.2] for details).

Corollary 2.19. The set $\overline{Y}_{\mathbf{w}}^{\mu,\tau}(\overline{\mathbb{F}})$ is nonempty if and only if $\mathbf{w} = (\tilde{w}_0, \tilde{w}_1, \dots, \tilde{w}_{f-1})$ where \tilde{w}_j is a $(2, 1, 0)$ -admissible element of \widetilde{W} .

One can describe $\mathrm{Adm}(2, 1, 0)$ quite concretely. Let \widetilde{W}^0 be the affine Weyl group of SL_3 . It is a Coxeter group generated by three reflections α, β and γ . The elements α and β which generate

the finite Weyl group we represent by

$$\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The element γ is given by

$$\gamma = \begin{pmatrix} 0 & 0 & v^{-1} \\ 0 & 1 & 0 \\ v & 0 & 0 \end{pmatrix}.$$

The subset we are interested in is a subset of the translation $v\widetilde{W}^0 \subset \widetilde{W}$ (i.e., those matrices with determinant $v^3(\text{unit})$).

Remark 2.20. The set $\text{Adm}(2, 1, 0)$ is defined to be the set of elements which are less than, or equal to, a translation t_λ where λ is a permutation of $(2, 1, 0)$. There are six extremal elements of length 4 corresponding to the six permutations of $(2, 1, 0)$. We divide the length three shapes into two different sets which behave differently. The set $\text{Adm}(2, 1, 0)$ is given in Table 1.

The following is the key result for classifying Kisin modules with \mathbb{F}' -coefficients, where \mathbb{F}' is a finite extension of \mathbb{F} .

Lemma 2.21. *Let $\mathfrak{M}, \mathfrak{M}' \in Y^{\mu, \tau}(\mathbb{F}')$ and let β, β' be eigenbases of $\mathfrak{M}, \mathfrak{M}'$ respectively. We define $A^{(j)} \stackrel{\text{def}}{=} \text{Mat}_\beta(\phi_{R, s_{j+1}(3)}^{(j)})$ (resp. $A'^{(j)} \stackrel{\text{def}}{=} \text{Mat}_{\beta'}(\phi_{R, s_{j+1}(3)}^{(j)})$) as in Definition 2.11. Assume further that there exists $J^{(j+1)} \in \mathcal{I}_1(\mathbb{F}')$ such that $A'^{(j)} = J^{(j+1)} A^{(j)}$ for all $0 \leq j \leq f-1$. Then there is an isomorphism $\mathfrak{M} \xrightarrow{\sim} \mathfrak{M}'$ in $Y^{\mu, \tau}(\mathbb{F}')$.*

Proof. We define, by induction, the following sequence $(J_n^{(j)})_{n \in \mathbb{N}}$ of elements in $\mathcal{I}_1(\mathbb{F}')$. Set $J_0^{(j)} \stackrel{\text{def}}{=} \text{Id}_3$ and, for $n \geq 1$, we set

$$J_{n+1}^{(j+1)} = J^{(j+1)} A^{(j)} \left(A^{(j)} s_{j+1, j} J_n^{(j), \varphi} s_{j+1, j}^{-1} \right)^{-1}$$

where we define $J_n^{(j), \varphi}$ as in Proposition 2.16 for any $J_n^{(j)} \in \mathcal{I}(F')$. Note that this defines a sequence in the pro- v Iwahori $\mathcal{I}_1(\mathbb{F}')$.

From the definition of $J_n^{(j)}$ and the hypothesis $A'^{(j)} = J^{(j+1)} A^j$ we obtain

$$(2.9) \quad A'^{(j)} = J_{n+1}^{(j+1)} A^j (s_{j+1, j} J_n^{(j), \varphi} s_{j+1, j}^{-1}).$$

Provided that the sequence $(J_n^{(j)})_n$ converges, we deduce the desired isomorphism $\mathfrak{M} \xrightarrow{\sim} \mathfrak{M}'$ via Proposition 2.15.

We now prove the convergence of the sequence $(J_n^{(j)})_n$. By the definition of the v -adic topology on $\mathcal{I}(\mathbb{F}')$, it is enough to prove that

$$(2.10) \quad v^{p(n-2)} | (J_{n+1}^{(j+1)} - J_n^{(j+1)})$$

for all $n \geq 3$. We induct on n .

For $n = 1$, we have $J_1^{(j)} = J^{(j)}$, and so $J_2^{(j+1)} - J_1^{(j+1)}$ equals

$$J^{(j+1)} A^{(j)} s_{j+1,j} \left(\text{Ad} \left(v^{a_{s_j(1),f-1-j}}, v^{a_{s_j(2),f-1-j}}, v^{a_{s_j(3),f-1-j}} \right) \varphi(J^{(j)} - \text{Id}_3) \right) s_{j+1,j}^{-1} (A^{(j)})^{-1}.$$

As $J^{(j)} \in \mathcal{I}_1(\mathbb{F}')$, we have

$$\varphi(J_1^{(j)} - \text{Id}_3) \in \begin{pmatrix} (v^p) & \mathbb{F}'[[v]] & \mathbb{F}'[[v]] \\ (v^p) & (v^p) & \mathbb{F}'[[v]] \\ (v^p) & (v^p) & (v^p) \end{pmatrix}.$$

By the genericity condition (2.1), we deduce that

$$v^3 | \text{Ad} \left(v^{a_{s_j(1),f-1-j}}, v^{a_{s_j(2),f-1-j}}, v^{a_{s_j(3),f-1-j}} \right) \cdot \varphi(J^{(j)} - \text{Id}_3).$$

The height condition gives us that $v^2 (A^{(j)})^{-1} \in \text{Mat}_3(\mathbb{F}'[[v]])$; hence $v | (J_2^{(j+1)} - J_1^{(j+1)})$. Since $v^p | \varphi(J_2^{(j)} - J_1^{(j)})$, we deduce by genericity condition that $v^2 | (J_3^{(j)} - J_2^{(j)})$. Finally, when $n = 3$, since $v^{2p} | \varphi(J_3^{(j)} - J_2^{(j)})$, we get $v^p | (J_4^{(j)} - J_3^{(j)})$.

By the inductive hypothesis, the genericity condition, and the height condition on $A^{(j)}$,

$$(2.11) \quad v^{p^2(n-2)-(p-4)-2} | \left(\text{Ad} \left(v^{a_{s_j(1),f-1-j}}, v^{a_{s_j(2),f-1-j}}, v^{a_{s_j(3),f-1-j}} \right) \left(\varphi(J_n^{(j)} - J_{n-1}^{(j)}) \right) \right) (A^{(j)})^{-1}.$$

For $n \geq 3$, we have

$$p^2(n-2) - p + 2 \geq p(p(n-2) - 1) \geq p(n-1)$$

since $p \geq 3$. □

With more careful analysis, Lemma 2.21 probably holds with weaker genericity conditions. However, we do not attempt such an analysis here.

Theorem 2.22. *Let \mathbb{F}' be a finite extension of \mathbb{F} . Let $\mathfrak{M} \in \overline{Y}_{\mathbf{w}}^{\mu,\tau}(\mathbb{F}')$ with $\mathbf{w} = (\tilde{w}_0, \tilde{w}_1, \dots, \tilde{w}_{f-1})$. There exists an eigenbasis β of \mathfrak{M} such that for each $j \in \mathbb{Z}/f\mathbb{Z}$ the matrix $A^{(j)} = \text{Mat}_{\beta}(\phi_{\mathfrak{M},s_{j+1}(3)}^{(j)})$ has the form given in the \tilde{w}_j -entry of Table 3.*

Proof. Let $A_1^{(j)} := \text{Mat}_{\beta_1^{(j)}}(\phi_{\mathfrak{M},s_{j+1}(3)}^{(j)})$ for some eigenbasis $\beta_1^{(j)}$ of $\mathfrak{M}^{(j)}$. If $A^{(j)} \in \mathcal{I}(\mathbb{F}') \tilde{w}_j \mathcal{I}(\mathbb{F}')$ is such that $A^{(j)}$ and $A_1^{(j)}$ lie in the same left coset $\mathcal{I}_1(\mathbb{F}') \setminus \mathcal{I}(\mathbb{F}') \tilde{w}_j \mathcal{I}(\mathbb{F}')$, then by Lemma 2.21 there exists an eigenbasis of $\beta^{(j)}$ of $\mathfrak{M}^{(j)}$ such that $A^{(j)} = \text{Mat}_{\beta^{(j)}}(\phi_{\mathfrak{M},s_{j+1}(3)}^{(j)})$.

In other words, by considering obvious isomorphism

$$\tilde{w}_j \cdot (P_{\tilde{w}_j} \setminus \mathcal{I}(\mathbb{F}')) \xrightarrow{\sim} \mathcal{I}_1(\mathbb{F}') \setminus \mathcal{I}(\mathbb{F}') \tilde{w}_j \mathcal{I}(\mathbb{F}')$$

where $P_{\tilde{w}_j} \stackrel{\text{def}}{=} (\tilde{w}_j^{-1} \mathcal{I}(\mathbb{F}') \tilde{w}_j) \cap \mathcal{I}_1(\mathbb{F}')$, we conclude that if $\mathfrak{M}^{(j)}$ has shape \tilde{w}_j , then there exists an eigenbasis β of \mathfrak{M} such that $\text{Mat}_\beta(\phi_{\mathfrak{M}, s_{j+1}(3)}^{(j)}) = A^{(j)}$ where $A^{(j)}$ is a representative of an element in $\tilde{w}_j \cdot (P_{\tilde{w}_j} \setminus \mathcal{I}(\mathbb{F}'))$. The statement follows now by an explicit casewise computation of the left coset $P_{\tilde{w}_j} \setminus \mathcal{I}(\mathbb{F}')$ for the various \tilde{w}_j in the $(2, 1, 0)$ -admissible set. \square

We now introduce the notion of a *gauge basis* of a mod p Kisin module:

Definition 2.23. Let $\overline{\mathfrak{M}} \in \overline{Y}_{\mathbf{w}}^{\mu, \tau}(\mathbb{F}')$. A *gauge basis* $\beta = (\beta^{(j)})_j$ of \mathfrak{M} is an eigenbasis such that for each $0 \leq j \leq f-1$, the matrix $\text{Mat}_\beta(\phi_{\mathfrak{M}, s_{j+1}(3)}^{(j)})$ is in the form given by the \tilde{w}_j -entry in Table 3.

In the above discussion, one could just as well have chosen to use $\phi_{\mathfrak{M}, s_{j+1}(2)}^{(j)}$ or $\phi_{\mathfrak{M}, s_{j+1}(1)}^{(j)}$ instead of $\phi_{\mathfrak{M}, s_{j+1}(3)}^{(j)}$. There is a simple way of determining the matrices for $\phi_{\mathfrak{M}, s_{j+1}(2)}^{(j)}$ or $\phi_{\mathfrak{M}, s_{j+1}(1)}^{(j)}$ in terms of $\phi_{\mathfrak{M}, s_{j+1}(3)}^{(j)}$. Furthermore, while the *shape* of the Kisin module depends on the choice of the isotypic piece, there is a simple recipe which relates them. As a consequence, we are able to reduce the number of shapes we have to consider from 25 to 9 using this cyclic symmetry.

Let

$$\delta = \begin{pmatrix} 0 & 0 & v^{-1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in \widetilde{W}.$$

Conjugation by δ induces an outer automorphism of \widetilde{W}^0 of order 3 satisfying

$$\delta \alpha \delta^{-1} = \beta, \quad \delta \beta \delta^{-1} = \gamma, \quad \delta \gamma \delta^{-1} = \alpha.$$

It is furthermore easy to check that

$$\delta \mathcal{I}(R) \delta^{-1} = \mathcal{I}(R)$$

for any \mathcal{O} -algebra R .

Proposition 2.24. Let $\mathfrak{M} \in Y^{[0, h], \tau}(R)$. Let $\beta^{(j)} = (f_1^{(j)}, f_2^{(j)}, f_3^{(j)})$ be an eigenbasis for \mathfrak{M} and let $A_{\mathbf{3}}^{(j)} \stackrel{\text{def}}{=} \text{Mat}_\beta(\phi_{\mathfrak{M}, s_j(3)}^{(j)})$ be as in Definition 2.11. Then

$$\left(\left\{ u^{e-\mathbf{a}_{s_j(2)}^{(j)} + \mathbf{a}_{s_j(3)}^{(j)}} \otimes f_{s_j(3)}^{(j-1)}, u^{\mathbf{a}_{s_j(1)}^{(j)} - \mathbf{a}_{s_j(2)}^{(j)}} \otimes f_{s_j(1)}^{(j-1)}, 1 \otimes f_{s_j(2)}^{(j-1)} \right\}, \left\{ u^{e-\mathbf{a}_{s_j(2)}^{(j)} + \mathbf{a}_{s_j(3)}^{(j)}} f_{s_j(3)}^{(j)}, u^{\mathbf{a}_{s_j(1)}^{(j)} - \mathbf{a}_{s_j(2)}^{(j)}} f_{s_j(1)}^{(j)}, f_{s_j(2)}^{(j)} \right\} \right) \\ \left(\left\{ u^{e-\mathbf{a}_{s_j(1)}^{(j)} + \mathbf{a}_{s_j(2)}^{(j)}} \otimes f_{s_j(2)}^{(j-1)}, u^{e-\mathbf{a}_{s_j(1)}^{(j)} + \mathbf{a}_{s_j(3)}^{(j)}} \otimes f_{s_j(3)}^{(j-1)}, 1 \otimes f_{s_j(1)}^{(j-1)} \right\}, \left\{ u^{e-\mathbf{a}_{s_j(1)}^{(j)} + \mathbf{a}_{s_j(2)}^{(j)}} f_{s_j(2)}^{(j)}, u^{e-\mathbf{a}_{s_j(1)}^{(j)} + \mathbf{a}_{s_j(3)}^{(j)}} f_{s_j(3)}^{(j)}, f_{s_j(1)}^{(j)} \right\} \right)$$

are bases for ${}^\varphi\mathfrak{M}_{s_j(2)}^{(j-1)}$ and $\mathfrak{M}_{s_j(2)}^{(j)}$ respectively (resp. for ${}^\varphi\mathfrak{M}_{s_j(1)}^{(j-1)}$ and $\mathfrak{M}_{s_j(1)}^{(j)}$ respectively). If $A_2^{(j)} \stackrel{\text{def}}{=} \text{Mat}_\beta(\phi_{\mathfrak{M}, \mathbf{a}_{s_{j+1}(2)}}^{(j)})$ and $A_1^{(j)} \stackrel{\text{def}}{=} \text{Mat}_\beta(\phi_{\mathfrak{M}, \mathbf{a}_{s_{j+1}(1)}}^{(j)})$ are defined in the evident way following Definition 2.11, then

$$A_2^{(j)} = \delta A_3^{(j)} \delta^{-1}, \quad A_1^{(j)} = \delta^2 A_3^{(j)} \delta^{-2}.$$

Proof. We give the proof for $\mathfrak{M}_{s_j(2)}^{(j-1)}$ as the other proof is similar. Let $C^{(j)} \stackrel{\text{def}}{=} \text{Mat}_\beta(\phi_{\mathfrak{M}}^{(j)})$ be as in Definition 2.11 so that by Proposition 2.13 we have

$$(2.12) \quad C^{(j-1)} = \text{Ad}_{s_j}(u^{\mathbf{a}_1}, u^{\mathbf{a}_2}, u^{\mathbf{a}_3})(A^{(j-1)}).$$

Let $c_1 = \beta\alpha$ be the permutation matrices corresponding to the cycle (132). It is clear that the elements listed in the statement form a basis for ${}^\varphi\mathfrak{M}_2^{(j-1)}$ and $\mathfrak{M}_2^{(j)}$ respectively and hence the same argument as in Proposition 2.13 shows that

$$(2.13) \quad A_2^{(j-1)} = \text{Ad}(u^{-\mathbf{a}_{s_j(3)}^{(j)} - e}, u^{-\mathbf{a}_{s_j(1)}^{(j)}}, u^{-\mathbf{a}_{s_j(2)}^{(j)}})(c_1^{-1} s_j^{-1} C^{(j-1)} s_j c_1),$$

Combining (2.12) and (2.13), we see that

$$A_2^{(j)} = \text{Ad}(\delta_2)(A_3^{(j)}),$$

where

$$\delta_2 = \text{Diag}(u^{-\mathbf{a}_{s_j(3)}^{(j)} - e}, u^{-\mathbf{a}_{s_j(1)}^{(j)}}, u^{-\mathbf{a}_{s_j(2)}^{(j)}}) c_2^{-1} \text{Diag}(u^{\mathbf{a}_{s_j(1)}^{(j)}}, u^{\mathbf{a}_{s_j(2)}^{(j)}}, u^{\mathbf{a}_{s_j(3)}^{(j)}}).$$

A direct computation shows that $\delta_2 = \delta$. □

Corollary 2.25. *Let $\mathfrak{M} \in Y^{[0, h], \tau}(\mathbb{F}')$. If \mathfrak{M} has shape $\mathbf{w} = (\tilde{w}_0, \dots, \tilde{w}_{f-1})$ then for any eigenbasis $\beta = (\beta^{(j)})$,*

$$\text{Mat}_\beta(\phi_{\mathfrak{M}, \mathbf{a}_{s_{j+1}(1)}}^{(j)}) \in \mathcal{I}(\mathbb{F}')(\delta \tilde{w}_j \delta^{-1}) \mathcal{I}(\mathbb{F}'), \quad \text{Mat}_\beta(\phi_{\mathfrak{M}, \mathbf{a}_{s_{j+1}(2)}}^{(j)}) \in \mathcal{I}(\mathbb{F}')(\delta^2 \tilde{w}_j \delta^{-2}) \mathcal{I}(\mathbb{F}')$$

for all $0 \leq j \leq f-1$.

Remark 2.26. As a consequence of Corollary 2.25, there is symmetry among the 25 shapes of $\text{Adm}(2, 1, 0)$. It is easy to see that $\delta \text{Adm}(2, 1, 0) \delta^{-1} = \text{Adm}(2, 1, 0)$ and that there are 9 orbits under conjugation by δ . In Table 3, we choose representatives for these 9 orbits and restrict our attention to those 9 shapes. One can deduce all our results for the remaining 18 shapes simply by conjugating by δ or δ^2 .

2.3. Étale φ -modules. We recall briefly some properties of étale φ -modules which are well-known. We refer to [CDMa, §2.1] and [CL, §5.3] for proofs.

Let $\mathcal{O}_{\mathcal{E},K}$ denote the p -adic completion of $\mathfrak{S}[\frac{1}{v}]$, where $\mathfrak{S} \stackrel{\text{def}}{=} W[[v]]$ endowed with the unique continuous Frobenius morphism such that the natural inclusion $\mathfrak{S}[\frac{1}{v}] \hookrightarrow \mathcal{O}_{\mathcal{E},K}$ is Frobenius-equivariant. Let R be a local, complete Noetherian \mathcal{O} -algebra. By base change, the ring $\mathcal{O}_{\mathcal{E},K} \widehat{\otimes}_{\mathbb{Z}_p} R$ is naturally endowed with a Frobenius endomorphism φ and we write $\Phi\text{-Mod}^{\text{ét}}(R)$ for the category of étale $(\varphi, \mathcal{O}_{\mathcal{E},K} \widehat{\otimes}_{\mathbb{Z}_p} R)$ -modules. We fix once and for all a sequence $\underline{p} \stackrel{\text{def}}{=} (p_n)_{n \in \mathbb{N}}$ where $p_n \in \overline{\mathbb{Q}_p}$ verify $p_{n+1}^p = p_n$ and $p_0 = -p$. We let $K_\infty \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{N}} K(p_n)$ and $G_{K_\infty} \stackrel{\text{def}}{=} \text{Gal}(\overline{\mathbb{Q}_p}/K_\infty)$.

By classical work of Fontaine ([Fon90]) we have an exact anti-equivalence of \otimes -categories:

$$\begin{aligned} \Phi\text{-Mod}^{\text{ét}}(R) &\xrightarrow{\sim} \text{Rep}_{G_{K_\infty}}(R) \\ \mathcal{M} &\longmapsto \mathbb{V}^*(\mathcal{M}) \stackrel{\text{def}}{=} \text{Hom}_{\Phi\text{-Mod}}(\mathcal{M}, \mathcal{O}_{\mathcal{E}^{un},K}) \end{aligned}$$

where $\mathcal{O}_{\mathcal{E}^{un},K}$ is the étale extension of $\mathcal{O}_{\mathcal{E}}$ corresponding to a separable closure of $k((v))$.

The above construction can also be carried out with descent datum. More precisely, choose $(\pi_n)_{n \in \mathbb{N}}$ to be the sequence satisfying $\pi_n^e = p_n$ and $\pi_{n+1}^p = \pi_n$ with $\pi_0 = \pi$. Then $L_\infty \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{N}} L(\pi_n)$ and $G_{L_\infty} \stackrel{\text{def}}{=} \text{Gal}(\overline{\mathbb{Q}_p}/L_\infty)$. We have $\text{Gal}(L_\infty/K_\infty) \cong \text{Gal}(L/K) = \Delta$. Let $\mathcal{O}_{\mathcal{E},L}$ denote the p -adic completion of $(W[[u]])[1/u]$ equipped with an action of $\text{Gal}(L_\infty/K_\infty) \cong \Delta$ characterized by $\widehat{g}(u) \stackrel{\text{def}}{=} \omega_\pi(g)u$. We define, in the evident way, the category $\Phi\text{-Mod}_{dd}^{\text{ét}}(R)$ of étale φ -module over $\mathcal{O}_{\mathcal{E},L} \widehat{\otimes}_{\mathbb{Z}_p} R$ with descent data.

We have an exact anti-equivalence of categories:

$$\begin{aligned} \Phi\text{-Mod}_{dd}^{\text{ét}}(R) &\xrightarrow{\sim} \text{Rep}_{G_{K_\infty}}(R) \\ \mathcal{M} &\longmapsto \mathbb{V}_{dd}^*(\mathcal{M}) \stackrel{\text{def}}{=} \text{Hom}_{\phi, \mathcal{O}_{\mathcal{E},L}}(\mathcal{M}, \mathcal{O}_{\mathcal{E}^{un},K}). \end{aligned}$$

Define T_{dd}^* to be the composition $Y^{[0,h],\tau}(R) \rightarrow \Phi\text{-Mod}_{dd}^{\text{ét}}(R) \rightarrow \text{Rep}_{G_{K_\infty}}(R)$, where the first map is given by tensoring with $\mathcal{O}_{\mathcal{E},L}$ (over $(W[[u]])$).

In order to compute T_{dd}^* , it is convenient to both remove the descent datum and pass to a single Frobenius. This is carried out in [CDMa, §2.1.3]. We briefly recall the construction.

For any \mathcal{O} -algebra R , we consider R as a W -algebra via σ_0 . We endow the ring $\mathcal{O}_{\mathcal{E},K} \widehat{\otimes}_{W,\sigma_0} R$ with a Frobenius φ^f (by base change). We can now define in the evident fashion the category $\Phi^f\text{-Mod}_{W,\sigma_0}^{\text{ét}}(R)$ of étale $(\varphi^f, \mathcal{O}_{\mathcal{E},K} \widehat{\otimes}_{W,\sigma_0} R)$ -modules and we have an exact equivalence of categories:

$$\begin{aligned} \Phi^f\text{-Mod}_{W,\sigma_0}^{\text{ét}}(R) &\xrightarrow{\sim} \text{Rep}_{G_{K_\infty}}(R) \\ \mathcal{M} &\longmapsto \mathbb{V}_W^*(\mathcal{M}) \stackrel{\text{def}}{=} \text{Hom}_{\Phi^f\text{-Mod}}(\mathcal{M}, \mathcal{O}_{\mathcal{E}^{un},K}). \end{aligned}$$

In particular, if $(\mathcal{M}, \phi_{\mathcal{M}}) \in \Phi\text{-Mod}^{\text{ét}}(R)$, then $(\mathcal{M}^{(0)}, \phi_{\mathcal{M}}^f) \in \Phi^f\text{-Mod}_{W, \sigma_0}^{\text{ét}}(R)$. This defines a functor $\varepsilon_0 : \Phi\text{-Mod}^{\text{ét}}(R) \rightarrow \Phi^f\text{-Mod}_{W, \sigma_0}^{\text{ét}}(R)$.

We have the following compatibility between the above constructions (cf. Theorem 2.1.6 and equation (12) in [CDMa]):

$$\begin{array}{ccccc}
 & & T_{dd}^* & & \\
 & \nearrow & & \searrow & \\
 Y^{[0,2],\tau}(R) & \xrightarrow{\quad} & \Phi\text{-Mod}_{dd}^{\text{ét}}(R) & \xrightarrow{\quad \mathbb{V}_{dd}^* \quad} & \text{Rep}_{G_{K_\infty}}(R) \\
 & \nwarrow & \uparrow & \nearrow & \\
 & & \Phi\text{-Mod}^{\text{ét}}(R) & \xrightarrow{\quad \varepsilon_0(\bullet) \quad} & \Phi^f\text{-Mod}_W^{\text{ét}}(R) \\
 & & \uparrow & & \uparrow \mathbb{V}_W^* \\
 & & -\otimes_{\mathcal{O}_{\mathcal{E},K}} \mathcal{O}_{\mathcal{E},L} & & (\bullet)^{\Delta=1} \quad \mathbb{V}^*
 \end{array}$$

If R is a \mathbb{F} -algebra and $\mathfrak{M} \in Y^{[0,2],\tau}(R)$, it will be useful to describe the étale $(\varphi^f, R((v)))$ -module $\varepsilon_0((\mathfrak{M} \otimes_{\mathbb{F}[[u]]} \mathbb{F}((u)))^{\Delta=1})$ explicitly in terms of the $A^{(j)}$.

Proposition 2.27. *Let $\mathfrak{M} \in Y^{[0,2],\tau}(R)$ and β be an eigenbasis of \mathfrak{M} . Write (s_j) for an orientation of τ , $(A^{(j)}) = \text{Mat}_{\beta}(\phi_{\mathfrak{M}, s_{j+1}(3)}^{(j)})$ and consider $\mathcal{M} = \mathfrak{M}[1/u] \in \Phi\text{-Mod}_{dd}^{\text{ét}}(R)$. Then the étale $(\varphi^f, R((v)))$ -module $\varepsilon_0(\mathcal{M}^{\Delta=1})$ is described with respect to the basis $\mathfrak{f} = (u^{\mathbf{a}_1^{(0)}} f_1^{(0)}, u^{\mathbf{a}_2^{(0)}} f_2^{(0)}, u^{\mathbf{a}_3^{(0)}} f_3^{(0)})$ by*

$$\text{Mat}_{\mathfrak{f}}(\phi_{\mathcal{M}^{(0)}}^f) = \prod_{j=0}^{f-1} s_{f-j} \cdot \varphi^{(j)} \left(A^{(f-1-j)} \begin{pmatrix} v^{a_{sf-j}(1),j} & 0 & 0 \\ 0 & v^{a_{sf-j}(2),j} & 0 \\ 0 & 0 & v^{a_{sf-j}(3),j} \end{pmatrix} \right) \cdot s_{f-j}^{-1}.$$

Proof. This is a direct computation (cf. [CDMa, (24)]). □

3. KISIN VARIETIES AND TANGENT SPACES

In this section, we show that under the genericity assumption (2.1) the Kisin variety is trivial, i.e., if $\bar{\rho} : G_K \rightarrow \mathrm{GL}_3(\mathbb{F}')$ comes from a Kisin module $\overline{\mathfrak{M}} \in Y^{\mu, \tau}(\mathbb{F}')$, then $\overline{\mathfrak{M}}$ is unique. As a consequence, one can attach a shape $\mathbf{w}(\bar{\rho}, \tau)$ to $\bar{\rho}$. Later, in Proposition 7.16 (also Table 9), we show that the shape $\mathbf{w}(\bar{\rho}, \tau)$ is closely related to the Serre weights of $\bar{\rho}$ which appear in the principal series type $\bar{\sigma}(\tau)$ (as was the case for GL_2 [Bre14]). In this section, we also show that the map on tangent spaces from deformations of Kisin modules to deformations of étale ϕ -modules is injective and that under mild assumptions the same is true for the restriction on Galois deformations from G_K to G_{K_∞} .

3.1. Kisin varieties. In what follows, let \mathbb{F}' denote a finite extension of \mathbb{F} . If $\mathcal{M} \in \Phi\text{-Mod}_{dd}^{\text{ét}}(\mathbb{F})$ is an étale $\mathcal{O}_{\mathcal{E}, L} \otimes_{\mathbb{Z}_p} \mathbb{F}$ -module with descent data (cf. §2.3) and R is any \mathbb{F}' -algebra, we set $\mathcal{M}_R \stackrel{\text{def}}{=} \mathcal{M} \otimes_{\mathbb{F}'((u))} R((u)) \in \Phi\text{-Mod}_{dd}^{\text{ét}}(R)$.

Definition 3.1. The *Kisin variety* $Y_{\mathcal{M}}^{\mu, \tau}$ of \mathcal{M} is a projective scheme over $\mathrm{Spec}(\mathbb{F})$ which represents the functor:

$$Y_{\mathcal{M}}^{\mu, \tau}(R) \stackrel{\text{def}}{=} \{\mathfrak{M}_R \subset \mathcal{M}_R \mid \mathfrak{M}_R[1/u] = \mathcal{M}_R, \phi_{\mathcal{M}_R}(\mathfrak{M}_R) \subset \mathfrak{M}_R, \mathfrak{M}_R \in Y^{\mu, \tau}(R)\}.$$

In other words, $Y_{\mathcal{M}}^{\mu, \tau}(R)$ is the set of $(k \otimes_{\mathbb{F}_p} R)[[u]]$ -lattices in \mathcal{M}_R which have type τ and satisfy height conditions. We can also consider

$$Y_{\mathcal{M}}^{[0, 2], \tau}(R) = \left\{ \mathfrak{M}_R \subset \mathcal{M}_R \mid \mathfrak{M}_R[1/u] = \mathcal{M}_R, \phi_{\mathcal{M}_R}(\mathfrak{M}_R) \subset \mathfrak{M}_R, \mathfrak{M}_R \in Y^{[0, 2], \tau}(R) \right\}$$

There is an obvious inclusion $Y_{\mathcal{M}}^{\mu, \tau} \subset Y_{\mathcal{M}}^{[0, 2], \tau}$. These are projective schemes because they are closed subschemes of finite type in the affine Grassmanian for the group $\mathrm{Res}_{k/\mathbb{F}} \mathrm{GL}_3$ (cf. [Kis09b, Proposition 2.1.7]).

Theorem 3.2. *If τ satisfies the genericity condition (2.1), then $Y_{\mathcal{M}}^{\mu, \tau}(\mathbb{F}')$ is either empty or a single point.*

Proof. We show that $Y_{\mathcal{M}}^{[0, 2], \tau}(\mathbb{F}')$ is either empty or a single point. Assume we have two $(k \otimes_{\mathbb{F}_p} \mathbb{F}')[[u]]$ -lattices \mathfrak{M}_1 and \mathfrak{M}_2 in $\mathcal{M}_{\mathbb{F}'}$. For $i \in \{1, 2\}$ choose an eigenbasis β_i for \mathfrak{M}_i , such that $\mathrm{Mat}_{\beta_i}(\phi_{\mathfrak{M}_i, s_{j+1}(3)}^{(j)}) = A_i^{(j)}$, where (s_j) denotes the orientation on τ . Let $(D^{(j)}) \in (\mathrm{GL}_3(\mathbb{F}'((u))))^f$ be the f -tuple of matrices which gives the basis for \mathfrak{M}_2 in terms of \mathfrak{M}_1 as in Proposition 2.15. Note that, *a priori*, the matrices $D^{(j)}$ have denominators in u .

We want to show that $D^{(j)} \in \mathrm{GL}_3(\mathbb{F}'[[u]])$ for all j . For all $j \in \mathbb{Z}/f\mathbb{Z}$, let us define $I^{(j)} = \mathrm{Ad}_{s_j}^{-1}(u^{\mathbf{a}^1}, u^{\mathbf{a}^2}, u^{\mathbf{a}^3})(D^{(j)}) \in \mathrm{GL}_3(\mathbb{F}'((v)))$ and recall the change of basis formula (2.4):

$$(3.1) \quad A_2^{(j)} = I^{(j+1)} A_1^{(j)} s_{j+1,j} \left(\mathrm{Ad} \left(v^{a_{s_j(1),f-j-1}}, v^{a_{s_j(2),f-j-1}}, v^{a_{s_j(3),f-j-1}} \right) (\varphi(I^{(j)})^{-1}) \right) s_{j+1,j}^{-1}.$$

Write $I^{(j+1)} = v^{-k_{j+1}} I^{(j+1),+}$ where $k_{j+1} \in \mathbb{Z}$ and $I^{(j+1),+} \in \mathrm{Mat}_3(\mathbb{F}'[[v]])$ is such that $I^{(j+1),+} \not\equiv 0 \pmod{v}$. Similarly, write $I^{(j)} = v^{-k_j} I^{(j),+}$. Rearranging (3.1), we get

$$(3.2) \quad v^{-pk_j} \cdot s_{j+1,j} \left(\mathrm{Ad} \left(v^{a_{s_j(1),f-j-1}}, v^{a_{s_j(2),f-j-1}}, v^{a_{s_j(3),f-j-1}} \right) \cdot \varphi(I^{(j),+}) \right) s_{j+1,j}^{-1} = v^{-k_{j+1}} (A_2^{(j)})^{-1} I^{(j+1),+} A_1^{(j)}.$$

Multiplying through by $v^{2+k_{j+1}}$ the right side of (3.2) becomes integral. Since $(I^{(j),+})_{i,k} \in (\mathbb{F}')^\times + v\mathbb{F}'[[v]]$ for some $1 \leq i, k \leq 3$, $\mathrm{Ad} \left(v^{a_{s_j(1),f-j-1}}, v^{a_{s_j(2),f-j-1}}, v^{a_{s_j(3),f-j-1}} \right) (\varphi(I^{(j),+}))$ is at most divisible by $v^{a_{s_j(1),f-1-j} - a_{s_j(3),f-1-j}}$. We conclude that

$$(3.3) \quad k_{j+1} \geq pk_j - (a_{s_j(1),f-1-j} - a_{s_j(3),f-1-j}) - 2.$$

By the genericity assumption (2.1), $\max_j \{a_{s_j(1),f-1-j} - a_{s_j(3),f-1-j}\} < p-3$. Thus, if $k_j \geq 1$ for any j , then by iterating (3.3) we deduce that all k_j become arbitrary large. Thus, $k_j \leq 0$ for all j and $I^{(j)} \in \mathrm{Mat}_3(\mathbb{F}'[[v]])$. Taking the determinant of (3.1), we see immediately that $I^{(j)} \in \mathrm{GL}_3(\mathbb{F}'[[v]])$.

It remains to show that $I^{(j)} \in \mathcal{I}(\mathbb{F}')$: by Lemma 2.14, this is equivalent to $D^{(j)} \in \mathrm{GL}_3(\mathbb{F}'[[u]])$. Rearranging again the change of basis formula (2.4), we have

$$(3.4) \quad (A_2^{(j)})^{-1} I^{(j+1)} A_1^{(j)} = s_{j+1,j} \left(\mathrm{Ad} \left(v^{a_{s_j(1),f-j-1}}, v^{a_{s_j(2),f-j-1}}, v^{a_{s_j(3),f-j-1}} \right) (\varphi(I^{(j)})) \right) s_{j+1,j}^{-1}$$

so that, for $1 \leq k < h \leq 3$, we obtain

$$\left(\mathrm{Ad} \left(v^{a_{s_j(1),f-j-1}}, v^{a_{s_j(2),f-j-1}}, v^{a_{s_j(3),f-j-1}} \right) (\varphi(I^{(j)})) \right)_{hk} \in \left(v^{p\alpha_{hk} - (a_{s_j(k),f-1-j} - a_{s_j(h),f-1-j})} \right)$$

where the integers $\alpha_{hk} \in \mathbb{N}$ are defined by $(I^{(j)})_{h,k} \in (v^{\alpha_{h,k}})$. On the other hand, the height condition on $A_2^{(j)}$ forces the LHS in (3.4) to be an element in $\frac{1}{v^2} \mathrm{Mat}_3(\mathbb{F}'[[v]])$. In particular, we have $p\alpha_{hk} - (a_{s_j(k),f-1-j} - a_{s_j(h),f-1-j}) \geq -2$ for all $1 \leq k < h \leq 3$ and this implies $\alpha_{h,k} \geq 1$ for all $3 \geq h > k \geq 1$ by the genericity condition (2.1). Therefore, $I^{(j)} \in \mathcal{I}(\mathbb{F}')$, as required. \square

Theorem 3.2 allows us to attach a shape $\mathbf{w}(\bar{\rho}, \tau)$ to $\bar{\rho} : G_K \rightarrow \mathrm{GL}_3(\mathbb{F}')$ when the type τ is generic:

Definition 3.3. Let $\bar{\rho} : G_K \rightarrow \mathrm{GL}_3(\mathbb{F})$ and τ be as in Theorem 3.2. Assume there exists $\overline{\mathfrak{M}}_{\bar{\rho}} \in Y^{\mu, \tau}(\mathbb{F})$ such that $T_{dd}^*(\overline{\mathfrak{M}}_{\bar{\rho}}) \cong \bar{\rho}|_{G_{K\infty}}$. We define $\mathbf{w}(\bar{\rho}, \tau) \in \mathrm{Adm}(2, 1, 0)^f$ to be the shape of $\overline{\mathfrak{M}}_{\bar{\rho}}$.

Next, we study the tangent space at a closed point of $\overline{\mathfrak{M}} \in Y^{\mu, \tau}(\mathbb{F}')$. Since $\overline{\mathfrak{M}}$ often has automorphisms, we work at the category level. Define

$$\mathfrak{t}_{\overline{\mathfrak{M}}} = \left\{ (\mathfrak{M}, \delta_0) \mid \mathfrak{M} \in Y^{\mu, \tau}(\mathbb{F}'[\varepsilon]/\varepsilon^2), \delta_0 : \mathfrak{M}/\varepsilon\mathfrak{M} \xrightarrow{\sim} \overline{\mathfrak{M}} \right\}$$

which we consider as a category where morphisms are maps in $Y^{\mu, \tau}(\mathbb{F}'[\varepsilon]/\varepsilon^2)$ commuting with trivializations.

Proposition 3.4. *Assume that τ satisfies (2.1). The functor T_{dd}^* induces a fully faithful functor*

$$T_{\text{tan}}^* : \mathfrak{t}_{\overline{\mathfrak{M}}} \rightarrow \text{Rep}_{\mathbb{F}'[\varepsilon]/\varepsilon^2}(G_{K_\infty}).$$

Proof. The functor $\mathbb{V}_{dd}^* : \Phi\text{-Mod}_{dd}^{\text{ét}}(\mathbb{F}'[\varepsilon]/\varepsilon^2) \rightarrow \text{Rep}_{\mathbb{F}'[\varepsilon]/\varepsilon^2}(G_{K_\infty})$ is an anti-equivalence of categories so we are reduced to showing that

$$\mathfrak{M} \mapsto \mathfrak{M}[1/u]$$

is fully faithful for coefficients in $\mathbb{F}'[\varepsilon]/\varepsilon^2$. Let $\mathfrak{M}_1, \mathfrak{M}_2 \in \mathfrak{t}_{\overline{\mathfrak{M}}}$. Choose an eigenbasis $\overline{\beta}$ of $\overline{\mathfrak{M}}$ and let $A^{(j)} = \text{Mat}_{\overline{\beta}}(\phi_{\overline{\mathfrak{M}}, s_{j+1}(3)}^{(j)})$. For $i \in \{1, 2\}$, we fix eigenbases β_i of \mathfrak{M}_i lifting $\overline{\beta}$ and write $A^{(j)} + \varepsilon B_i^{(j)} = \text{Mat}_{\beta_i}(\phi_{\mathfrak{M}_i, s_{j+1}(3)}^{(j)})$ for some $B_i^{(j)} \in \text{Mat}_3(\mathbb{F}'[[v]])$. An isomorphism $\iota : \mathfrak{M}_1[1/u] \xrightarrow{\sim} \mathfrak{M}_2[1/u]$ which is trivial modulo (ε) satisfies $\text{Mat}_{\beta_1, \beta_2}(\iota) = \text{id}_3 + \varepsilon D^{(j)}$ for some $D^{(j)} \in \text{Mat}_3(\mathbb{F}'((u)))$.

Define $Y^{(j)} \stackrel{\text{def}}{=} \text{Ad}_{s_j}^{-1}(u^{\mathbf{a}_1}, u^{\mathbf{a}_2}, u^{\mathbf{a}_3})(D^{(j)}) \in \text{Mat}_3(\mathbb{F}'((v)))$. A direct computation using Proposition 2.15 gives

$$B_2^{(j)} = B_1^{(j)} + Y^{(j+1)} A^{(j)} - A^{(j)} s_{j+1, j} \left(\text{Ad} \left(v^{a_{s_j(1), f-j-1}}, v^{a_{s_j(2), f-j-1}}, v^{a_{s_j(3), f-j-1}} \right) (\varphi(Y^{(j)})) \right) s_{j+1, j}^{-1}.$$

Arguing as in Theorem 3.2, we deduce that $Y^{(j)} \in \text{Mat}_3(\mathbb{F}'[[v]])$. More precisely, for $i \in \{j, j+1\}$, we let $k_i \in \mathbb{N}$ be defined by $v^{k_i} Y^{(i)} = Y^{(i), +} \in \text{Mat}(\mathbb{F}'[[v]])$, where $Y^{(i), +} \not\equiv 0$ modulo v . We deduce as in Theorem 3.2, that

$$(3.5) \quad v^{-pk_j} s_{j+1, j} \left(\text{Ad} \left(v^{a_{s_j(1), f-j-1}}, v^{a_{s_j(2), f-j-1}}, v^{a_{s_j(3), f-j-1}} \right) (\varphi(Y^{(j), +})) \right) s_{j+1, j}^{-1} = \\ = (A_1^{(j)})^{-1} \left(-B_2^{(j)} + v^{-k_{j+1}} Y^{(j+1), +} A_1^{(j)} + B_1^{(j)} \right)$$

and therefore, by the height condition on $A_1^{(j)}$:

$$v^{2-pk_j} \left(\text{Ad} \left(v^{a_{s_j(1), f-j-1}}, v^{a_{s_j(2), f-j-1}}, v^{a_{s_j(3), f-j-1}} \right) (\varphi(Y^{(j), +})) \right) \in \text{Mat}_3(\mathbb{F}'[[v]]).$$

We again obtain the key inequality (3.3) which forces $k_j \leq 0$ for all j .

It remains to show that $D^{(j)} \in \text{Mat}_3(\mathbb{F}'[[u]])$, which is equivalent to proving that $Y^{(j)}$ is upper triangular modulo v . From (3.5), specialized at $k_j = 0$ and $Y^{(j), +} = Y^{(j)}$, we see that $v^2 \text{Ad} \left(v^{a_{s_j(1), f-j-1}}, v^{a_{s_j(2), f-j-1}}, v^{a_{s_j(3), f-j-1}} \right) (\varphi(Y^{(j)}))$ is integral. By the genericity assumption, the same argument in the proof of Theorem 3.2 shows that $Y^{(j)}$ is upper triangular mod v . \square

3.2. Kisin resolution. We now apply the computations from the previous section to obtain preliminary results in our study of potentially crystalline deformation rings. Fix a representation $\bar{\rho} : G_K \rightarrow \mathrm{GL}_3(\mathbb{F})$.

Let $R_{\bar{\rho}}^{\mu, \tau}$ be the framed potentially crystalline deformation ring with parallel Hodge-Tate weights $(2, 1, 0)$ and inertial type τ as in [Kis08]. Consider a variation which includes descent datum of the resolution of the deformation ring introduced in [Kis08, § (1.4)]. Let

$$\Theta : Y_{\bar{\rho}}^{\mu, \tau} \rightarrow \mathrm{Spec} R_{\bar{\rho}}^{\mu, \tau}$$

be the projective morphism defined in section §5.3 of [CL] which is “the moduli of Kisin modules with descent” over $\mathrm{Spec} R_{\bar{\rho}}^{\mu, \tau}$.

Since we always work in parallel weight $(2, 1, 0)$, we drop μ from the notation. Set $D_{\bar{\rho}}^{\square, \tau} \stackrel{\mathrm{def}}{=} \mathrm{Spf} R_{\bar{\rho}}^{\square, \tau}$. For τ generic, the fiber of Θ over $\bar{\rho}$ is a point (Theorem 3.2), hence $Y_{\bar{\rho}}^{\mu, \tau} = \mathrm{Spec} R_{\bar{\rho}}^{\tau, \square}$ where $R_{\bar{\rho}}^{\tau, \square}$ is the complete local \mathcal{O} -algebra representing the deformation problem

$$(3.6) \quad D_{\bar{\rho}}^{\tau, \square}(A) \stackrel{\mathrm{def}}{=} \{(\mathfrak{M}_A, \rho_A, \delta_A) \mid \mathfrak{M}_A \in Y^{\mu, \tau}(A), \rho_A \in D_{\bar{\rho}}^{\square, \tau}(A), \delta_A : T_{dd}^*(\mathfrak{M}_A) \cong (\rho_A)|_{G_{K_\infty}}\}.$$

Furthermore, $\mathrm{Spec} R_{\bar{\rho}}^{\tau, \square}$ is finite over $\mathrm{Spec} R_{\bar{\rho}}^{\mu, \tau}$.

Corollary 3.5. *Let $\bar{\rho} : G_K \rightarrow \mathrm{GL}_3(\mathbb{F})$. If τ is generic, then*

$$\Theta : Y_{\bar{\rho}}^{\mu, \tau} \rightarrow \mathrm{Spec} R_{\bar{\rho}}^{\mu, \tau}$$

is an isomorphism.

Proof. We saw above that Θ is a finite morphism. By [CL, Theorem 5.20], $\Theta[1/p]$ is an isomorphism. Proposition 3.4 implies that the induced map

$$\mathrm{Spf} R_{\bar{\rho}}^{\tau, \square} \rightarrow \mathrm{Spf} R_{\bar{\rho}}^{\mu, \tau}$$

is an injective on tangent spaces. Hence Θ is a closed immersion. Since $R_{\bar{\rho}}^{\mu, \tau}$ is \mathcal{O} -flat, we conclude that Θ is an isomorphism. \square

3.3. Galois cohomology. In this section, we work with Galois cohomology to prove that -under mild hypotheses- the natural restriction map from G_K -deformations to G_{K_∞} -deformations is a closed immersion. This will be important in §5.2.

Definition 3.6. Let $\bar{\rho} : G_K \rightarrow \mathrm{GL}_3(\mathbb{F})$ be a continuous semisimple Galois representation and let $m \in \mathbb{N}$ be an integer. Let $\mathbf{a}_k = (a_{k,j})_j$ with $0 \leq a_{k,j} \leq p-1$ and $j \in \mathbb{Z}/f\mathbb{Z}$ be f -tuples such that $\bar{\rho}|_{I_K} \cong \omega_f^{\mathbf{a}_1^{(0)}} \oplus \omega_f^{\mathbf{a}_2^{(0)}} \oplus \omega_f^{\mathbf{a}_3^{(0)}}$. We say that $\bar{\rho}$ is m -generic if

$$m \leq |a_{1,j} - a_{2,j}|, |a_{2,j} - a_{3,j}|, |a_{1,j} - a_{3,j}| \leq p-1-m$$

for all j . We say that a continuous Galois representation $\bar{\rho} : G_K \rightarrow \mathrm{GL}_3(\mathbb{F})$ is m -generic if there exists a finite unramified extension K'/K such that $\bar{\rho}^{ss}|_{I_{K'}}$ is the direct sum of characters and is m -generic in the previous sense. This does not depend on the extension K'/K .

There is a weaker genericity condition which suffices for our Galois cohomology arguments:

Definition 3.7. Let $\bar{\rho} : G_K \rightarrow \mathrm{GL}_n(\mathbb{F})$ be a continuous Galois representation. We say $\bar{\rho}$ is *cyclotomic free* if $\bar{\rho}$ becomes ordinary over an unramified extension K'/K of degree prime to p such that

$$H^0(G_{K'}, (\bar{\rho}|_{G_{K'}}^{ss}) \otimes \omega^{-1}) = 0.$$

Proposition 3.8. *If $p > 3$ and $\bar{\rho} : G_K \rightarrow \mathrm{GL}_3(\mathbb{F})$ is 2-generic, then $\mathrm{ad}(\bar{\rho})$ is cyclotomic free.*

Proof. Let K' denote the unramified extension of K of degree 6. Then $\bar{\rho}|_{G_{K'}}$ is ordinary and we can write $(\bar{\rho}|_{G_{K'}})^{ss}|_{I_{K'}} = \bigoplus_{i=1}^3 \omega_{6f}^{\mathbf{a}_i^{(0)}}$ where the $6f$ -tuple $\mathbf{a}_i \in \{0, 1, \dots, p-1\}^{6f}$ is 2-generic. In particular, it follows that $\mathrm{ad}(\bar{\rho})|_{G_{K'}} = \mathrm{ad}(\bar{\rho}|_{G_{K'}})$ is ordinary with diagonal characters of the form $\omega_{6f}^{\mathbf{a}_i^{(0)} - \mathbf{a}_{i'}^{(0)}}$, where $i, i' \in \{1, 2, 3\}$. Its semisimplification does not have cyclotomic constituents as long as $\mathbf{a}_i^{(0)} - \mathbf{a}_{i'}^{(0)} \not\equiv 1 + p + \dots + p^{6f-1} \pmod{p^{6f} - 1}$ which follows easily from the 2-genericity assumption. \square

Lemma 3.9. *Let $\bar{\rho} : G_K \rightarrow \mathrm{GL}_n(\mathbb{F})$ be cyclotomic free. Then the restriction map $H^1(K, \bar{\rho}) \rightarrow H^1(K_\infty, \bar{\rho})$ is injective.*

Proof. We first assume that $\bar{\rho}$ is ordinary. In this case the proof is a standard dévissage. More precisely, we have an exact sequence $0 \rightarrow \bar{\rho}_1 \rightarrow \bar{\rho} \rightarrow \bar{\chi} \rightarrow 0$ where $\bar{\chi} : G_K \rightarrow \mathbb{F}^\times$ is not the cyclotomic character and $\bar{\rho}_1 : G_K \rightarrow \mathrm{GL}_{n-1}(\mathbb{F})$ is ordinary (and $\bar{\rho}_1|_{G_{K'}}^{ss}$ does not contain the cyclotomic character).

Group cohomology provides us with the following commutative diagram, with exact rows:

$$\begin{array}{ccccccc} H^0(K, \bar{\chi}) & \xrightarrow{\delta} & H^1(K, \bar{\rho}_1) & \longrightarrow & H^1(K, \bar{\rho}) & \longrightarrow & H^1(K, \bar{\chi}) \\ \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 \\ H^0(K_\infty, \bar{\chi}) & \xrightarrow{\delta} & H^1(K_\infty, \bar{\rho}_1) & \longrightarrow & H^1(K_\infty, \bar{\rho}) & \longrightarrow & H^1(K_\infty, \bar{\chi}) \end{array}$$

the vertical maps being induced by restriction to G_{K_∞} . By [GLS15, Lemma 5.4.2], the morphism f_3 is injective. By dévissage we can assume that f_1 is injective. Finally, as $\bar{\chi}$ is a character, f_0 is surjective. Hence f_2 is injective by the “four lemma.”

As for the general case, we have an exact sequence of groups

$$\begin{array}{ccccccc} 1 & \longrightarrow & G_{K'} & \xrightarrow{\triangleleft} & G_K & \longrightarrow & \mathrm{Gal}(K'/K) \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \uparrow \wr \\ 1 & \longrightarrow & G_{K'_\infty} & \xrightarrow{\triangleleft} & G_{K_\infty} & \longrightarrow & \mathrm{Gal}(K'_\infty/K_\infty) \longrightarrow 1 \end{array}$$

and hence restriction to G_{K_∞} induces a morphism between the Hochschild-Serre spectral sequences

$$(3.7) \quad \begin{array}{ccc} H^r(\mathrm{Gal}(K'/K), H^s(K', \bar{\rho})) & \Longrightarrow & H^{r+s}(K, \bar{\rho}) \\ \downarrow & & \downarrow \\ H^r(\mathrm{Gal}(K'_\infty/K_\infty), H^s(K'_\infty, \bar{\rho})) & \Longrightarrow & H^{r+s}(K_\infty, \bar{\rho}). \end{array}$$

As $p \nmid [K' : K]$, the category of $\mathrm{Gal}(K'/K)$ -representations over \mathbb{F} is semisimple and the above spectral sequence becomes simply

$$(3.8) \quad \begin{array}{ccc} H^0(\mathrm{Gal}(K'/K), H^s(K', \bar{\rho})) & \xrightarrow{\sim} & H^s(K, \bar{\rho}) \\ \downarrow & & \downarrow \\ H^0(\mathrm{Gal}(K'_\infty/K_\infty), H^s(K'_\infty, \bar{\rho})) & \xrightarrow{\sim} & H^s(K_\infty, \bar{\rho}). \end{array}$$

The conclusion follows from the result in the ordinary case. \square

A statement similar to Lemma 3.9 (via a slightly different argument) has been obtained in Proposition 6.1 [Gao].

A similar argument yields the following:

Lemma 3.10. *Let $\bar{\rho} : G_K \rightarrow \mathrm{GL}_n(\mathbb{F})$ be cyclotomic free. Then the natural restriction map $B^1(K, \bar{\rho}) \rightarrow B^1(K_\infty, \bar{\rho})$ is injective.*

Proof. The argument follows closely the proof of Lemma 3.9 above. Let $V_{\bar{\rho}}$ be the \mathbb{F} -linear space underlying $\bar{\rho}$. Then one has

$$\frac{V_{\bar{\rho}}}{(V_{\bar{\rho}})^{G_K}} \xrightarrow{\sim} B^1(K, \bar{\rho}), \quad \frac{V_{\bar{\rho}}}{(V_{\bar{\rho}})^{G_{K_\infty}}} \xrightarrow{\sim} B^1(K_\infty, \bar{\rho});$$

therefore it is enough to prove that the restriction map $H^0(K, \bar{\rho}) \rightarrow H^0(K_\infty, \bar{\rho})$ is surjective.

We assume first that $\bar{\rho}$ is ordinary. Let us fix an extension $0 \rightarrow \bar{\rho}_1 \rightarrow \bar{\rho} \rightarrow \bar{\chi} \rightarrow 0$, where $(\bar{\rho}_1)^{\mathrm{ss}}$ and the character $\bar{\chi} : G_K \rightarrow \mathbb{F}^\times$ do not have cyclotomic constituents. The restriction functor to

G_{K_∞} and classical group cohomology give us the following commutative diagram, with exact lines:

$$\begin{array}{ccccccc}
 H^0(K, \bar{\rho}_1) & \longrightarrow & H^0(K, \bar{\rho}) & \longrightarrow & H^0(K, \bar{\chi}) & \xrightarrow{\delta} & H^1(K, \bar{\rho}_1) \\
 \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 \\
 H^0(K_\infty, \bar{\rho}_1) & \longrightarrow & H^0(K_\infty, \bar{\rho}) & \longrightarrow & H^0(K_\infty, \bar{\chi}) & \xrightarrow{\delta} & H^1(K_\infty, \bar{\rho}_1).
 \end{array}$$

By Lemma 3.9 the morphism f_3 is injective; for $i \in \{0, 1, 2\}$ the morphisms f_i are obviously injective. If f_0, f_2 are both surjective, the “four lemma” again shows that f_1 is surjective as well. Therefore, by dévissage, it is enough to show that $H^0(K_\infty, \bar{\chi}) \neq 0$ if and only if $\bar{\chi}$ is the trivial character of G_K . This is immediate as all characters are tame.

The deduction for the general case is formal: as in the proof of Lemma 3.9, the hypotheses on $[K' : K]$ give us the commutative diagram (3.8). Again, as the natural map $G_{K_\infty}/G_{K'_\infty} \rightarrow G_K/G_{K'}$ is an isomorphism, the isomorphism $H^0(K', \bar{\rho}) \xrightarrow{\sim} H^0(K'_\infty, \bar{\rho})$ obtained in the ordinary case respects the residual Galois action on both sides. Therefore $H^0((\text{Gal}(K'/K), H^0(K', \bar{\rho}))) \hookrightarrow H^0((\text{Gal}(K'_\infty/K_\infty), H^0(K'_\infty, \bar{\rho})))$ is an isomorphism. \square

4. FINITE HEIGHT K_∞ -DEFORMATIONS

Let $\overline{\mathfrak{M}} \in Y^{\mu, \tau}(\mathbb{F})$ with a basis $\overline{\beta}$ as in Theorem 2.22. We will now compute the deformations of $(\overline{\mathfrak{M}}, \overline{\beta})$ according to the shape of $\overline{\mathfrak{M}}$. Roughly, we are giving local coordinates for $Y^{\mu, \tau}$ at $\overline{\mathfrak{M}}$. This amounts to giving coordinates for the Pappas-Zhu local model $M(\mu)$ discussed after Definition 2.17, though this won't be used. The strategy will be to start with an arbitrary lift of $\overline{\mathfrak{M}}$ to a local Noetherian \mathcal{O} -algebra R with finite residue field and then by a convergence process put the Frobenius into a special form where entries are polynomials with coefficients in R with controlled degree. In this special form, it is straightforward to impose the height $[0, 2]$ condition as well as a determinant condition. The algorithm combines the u -adic and max adic topologies in suitable way. For GL_2 , a similar strategy was introduced in setting of Breuil modules in [Bre14] and was implemented for Kisin modules for non-generic types in [CDMa].

Theorem 4.1. *Let R be a complete local Noetherian \mathcal{O} -algebra with finite residue field \mathbb{F} and write $(s_j)_j \in S_3^f$ for the orientation of τ . Let $\mathfrak{M} \in Y^{\mu, \tau}(R)$ with $\overline{\mathfrak{M}} := \mathfrak{M} \otimes_R \mathbb{F}$ of shape $\mathbf{w} = (\tilde{w}_0, \tilde{w}_1, \dots, \tilde{w}_{f-1})$. Then there exists an eigenbasis β for \mathfrak{M} such that for each $0 \leq j \leq f-1$ the matrix $\tilde{A}^{(j)} = \mathrm{Mat}_\beta(\phi_{\mathfrak{M}, s_{j+1}(3)}^{(j)})$ has the form given in row \tilde{w}_j in Table 4.*

4.1. Algorithm. Let R be a complete local Noetherian \mathcal{O} -algebra with maximal ideal m_R and residue field $R/m_R \cong \mathbb{F}$. Let $P \in R[[u]]$. For any $r \in R$, let $v_R(r) = \max\{i \in \mathbb{N} \mid k \geq 0, r \in m_R^k\}$. This is finite unless $r = 0$, by Krull's intersection theorem.

Definition 4.2. Let $P = \sum_i r_i v^i \in R[[v]]$. Define

$$d_R(P) = \min_i 3v_R(r_i) + i.$$

Proposition 4.3. *For any $P, Q \in R[[v]]$ and any $l \in \mathbb{N}$, we have*

- (1) $d_R(P + Q) \geq \min(d_R(P), d_R(Q))$;
- (2) $d_R(PQ) \geq d_R(P) + d_R(Q)$;
- (3) $d_R(\mathrm{Tr}_l(P)) \geq d_R(P)$

where $\mathrm{Tr}_l : R[[v]] \rightarrow R[[v]]$ is the order v^l -truncation map, defined by $\mathrm{Tr}_l(\sum_i r_i v^i) \stackrel{\mathrm{def}}{=} \sum_{i \geq l+1} r_i v^i$.

Proof. This follows from $v_R(ab) \geq v_R(a) + v_R(b)$ and $v_R(a+b) \geq \min(v_R(a), v_R(b))$ for $a, b \in R$. \square

The algorithm proceeds by successive row operations which we introduce now. For any $x \in R[[v]]$, we define

$$U_{12}(x) := \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, U_{13}(x) := \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, U_{23}(x) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, for any $x \in R[[v]]$, we define

$$L_{21}(x) := \begin{pmatrix} 1 & 0 & 0 \\ vx & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, L_{31}(x) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ vx & 0 & 1 \end{pmatrix}, L_{32}(x) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & vx & 1 \end{pmatrix}.$$

$$D_{11}(x) := \begin{pmatrix} 1+vx & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D_{22}(x) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+vx & 0 \\ 0 & 0 & 1 \end{pmatrix}, D_{33}(x) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+vx \end{pmatrix}.$$

Given a matrix $M = (P_{ik}) \in \text{Mat}_3(R[[v]])$, we define

$$d_R(M)_{ik} = d_R(P_{ik}) \text{ and } d_R(M) = \min_{i,k} \{d_R(M)_{ik}\}.$$

The essence of the algorithm is as follows. Let $\beta_n = (\beta_n^{(j)})$ be an eigenbasis for \mathfrak{M}_R at the n -th step of the algorithm. If $A_n^{(j)} = \text{Mat}_{\beta_n}(\phi_{\mathfrak{M}, s_{j+1}(3)}^{(j)})$, we can always write

$$A_n^{(j)} = B_{\tilde{w}_j, n}^{(j)} + E_n^{(j)}$$

where the entries of $B_{\tilde{w}_j, n}^{(j)} \in \text{Mat}_3(R[[v]])$ verify the degree conditions described in Table 4 according to the shape \tilde{w}_j of $\overline{\mathfrak{M}}_R^{(j)}$. We call $E_n^{(j)}$ the *error term* associated to β_n . The inductive step is to show that there exists a new eigenbasis β_{n+1} with error $E_{n+1}^{(j)}$ such that

$$d_R(E_{n+1}^{(j)}) > d_R(E_n^{(j)}).$$

We start with a few definitions.

Definition 4.4. Let $A^{(j)} \in \text{Mat}_3(R[[v]])$ and $\tilde{w}_j \in \widetilde{W}$ be the shape of $\overline{\mathfrak{M}}$ at j . Then there exists a unique decomposition in $\text{Mat}_3(R[[v]])$

$$A^{(j)} = B_{\tilde{w}_j}^{(j)} + E^{(j)}$$

such that for all i, k one has

$$\deg_v((B_{\tilde{w}_j}^{(j)})_{ik}) = (\deg(\tilde{A}_{\tilde{w}_j}^{(j)}))_{ik} < \deg_v(E_{ik}^{(j)})$$

where $(\deg(\tilde{A}_{\tilde{w}_j}^{(j)}))_{ik} \in \{-\infty, 0, 1, 2\}$ is defined in the middle column of Table 4.

The *defect* of $A^{(j)}$ at the entry (ik) is defined as $\delta(A^{(j)})_{ik} \stackrel{\text{def}}{=} d_R(E^{(j)})_{ik}$. Similarly, the *total defect* of $A^{(j)}$ is defined as

$$\delta(A^{(j)}) \stackrel{\text{def}}{=} d_R(E^{(j)}).$$

Typically, the matrix $A^{(j)}$ in definition 4.4 is either $\text{Mat}_\beta(\phi_{\mathfrak{M}, s_{j+1}(3)}^{(j)})$ in some eigenbasis β on \mathfrak{M}_R , or its modification by row and (adjoint)-column operations by $U_{ik}(x^{(j)})$, $L_{ik}(x^{(j)})$ (cf. Proposition 4.5, Proposition 4.9).

Let $(x^{(j)})$ denote an f -tuple of elements of $R[[v]]$. An *elementary operation* (associated to $(x^{(j)})$) is a change of eigenbasis on \mathfrak{M}_R such that for each embedding j , the associated matrix $D^{(j)}$ as in Proposition 2.15 can be written as $D^{(j)} = \text{Ad}_{s_j}(u^{\mathbf{a}_1}, u^{\mathbf{a}_2}, u^{\mathbf{a}_3})(I^{(j)})$ for some $I^{(j)} \in \{U_{ik}(x^{(j)}), L_{ki}(x^{(j)}), i > k, D_{ii}(x^{(j)})\}$.

The following proposition shows that the right multiplication by $s_{j+1}^{-1}s_j I^{(j),\varphi} s_j^{-1}s_{j+1}$ in (2.4) does not lose any precision when $I^{(j)}$ is an elementary operation.

Proposition 4.5. *Let $\mathfrak{M} \in Y^{[0,2],\tau}(R)$, with eigenbasis β , and let $A^{(j)} = B_{\tilde{w}_j}^{(j)} + E^{(j)}$ as in Definition 4.4, where we have set $A^{(j)} \stackrel{\text{def}}{=} \text{Mat}_\beta(\phi_{\mathfrak{M}, s_{j+1}(3)}^{(j)})$. Let $I^{(j)}$ be an elementary operation associated to $x^{(j)}$, with $d_R(x^{(j)}) \geq \delta(A^{(j)}) - 3$ and define*

$$I^{(j),\varphi} \stackrel{\text{def}}{=} \text{Ad}(v^{a_{s_j(1),f-1-j}}, v^{a_{s_j(2),f-1-j}}, v^{a_{s_j(3),f-1-j}}) \cdot \varphi(I^{(j)})^{-1}.$$

Then one has

$$A^{(j)} s_{j+1}^{-1} s_j I^{(j),\varphi} s_j^{-1} s_{j+1} = B_{\tilde{w}_j}^{(j)} + E'^{(j)}$$

where $d_R(E'^{(j)}) \geq d_R(E^{(j)})$.

Moreover, if $d_R(x^{(j)}) \geq \delta(A^{(j)}) - 2$ then $d_R(E'^{(j)} - E^{(j)}) \geq \delta(A^{(j)}) + 1$.

In particular, in the notation of Proposition 2.15, if two eigenbases β_1, β_2 are related by an elementary operation, then one has $\delta(A_2^{(j)}) > \delta(A_1^{(j)})$ if $I^{(j+1)} A_1^{(j)} = B_{\tilde{w}_j}^{(j)} + E_2^{(j)}$ with $d_R(E_2^{(j)}) > \delta(A_1^{(j)})$.

Proof. We saw in Proposition 2.16 that

$$I^{(j),\varphi} = \text{id} + v^3 X$$

where $X \in \text{Mat}(R[[v]])$. The same calculation shows that $d_R(X) \geq d_R(x^{(j)})$. Therefore

$$A^{(j)} \cdot s_{j+1,j} \cdot I^{(j),\varphi} \cdot s_{j+1,j}^{-1} = B_{\tilde{w}_j}^{(j)} + E^{(j)} \left(\text{id} + v^3 s_{j+1,j} X s_{j+1,j}^{-1} \right) + v^3 B_{\tilde{w}_j}^{(j)} s_{j+1,j} X s_{j+1,j}^{-1}$$

and, under the assumptions on $x^{(j)}$, we have $d_R(v^3 B_{\tilde{w}_j}^{(j)} X) \geq d(A^{(j)})$.

The last statement follows from the change of basis formula (2.4) in Proposition 2.15. \square

The previous proposition can be summarized as follows: in order to increase the defect of $A^{(j)}$ via elementary operations, it is enough to perform elementary row operations on $A^{(j)}$ via the matrices $U_{ik}(x)$, $L_{ik}(x)$, $D_{ii}(x)$.

We introduce the crucial notion of *pivots* associated to a shape:

Definition 4.6. Let $\tilde{w}_j \in \tilde{W}$ be the shape at j of $\overline{\mathfrak{M}} \in Y^{[0,2],\tau}(\mathbb{F})$. The *pivots* of \tilde{w}_j are the pairs (m, k) such that the (m, k) -entry of $\tilde{w}_j \in N_{\mathrm{GL}_3}(T)(\mathbb{F}((v)))$ is non-zero.

Let $\mathfrak{M} \in Y^{[0,2],\tau}(R)$ be a Kisin module and let \tilde{w}_j be the shape of $\overline{\mathfrak{M}} \stackrel{\mathrm{def}}{=} \mathfrak{M} \otimes_R \mathbb{F}$ at j . If $A^{(j)} = \mathrm{Mat}_{\beta}(\phi_{\mathfrak{M}, s_{j+1}(3)}^{(j)})$ with respect an eigenbasis β , we say that the pair $(m, k) \in \{1, 2, 3\}^2$ is a *pivot* of $A^{(j)}$ if (m, k) is a pivot of the shape \tilde{w}_j . We define the *degree* of the pivot (m, k) to be $\deg_v(\overline{A}_{mk}^{(j)})$.

Remark 4.7. The degree conditions in the third column of Table 4 are determined by the degree and location of the pivots. Roughly, Proposition 4.5 allows row operations by matrices in the Iwahori subgroup. If (m, k) is a pivot of degree i , then any entry above (m, k) is of degree strictly less than i and any entry below (m, k) is of degree less than i . There also always the condition that entries below the diagonal are divisible by v .

The key lemma that enables us to control the convergence is the following:

Lemma 4.8. *Keep the notation as in Definition 4.6. Assume that the eigenbasis β lifts a gauge basis $\overline{\beta}$ of $\overline{\mathfrak{M}}_R$ (cf. Definition 2.23; in particular $\overline{A}^{(j)} = \overline{A}_{\tilde{w}_j}^{(j)}$ as in Table 3).*

Let (m, k) be a pivot for $A^{(j)}$ of degree i . For all $k' \geq k$, we can write

$$A_{mk'}^{(j)} = v^i P_{mk'} + Q_{mk'}$$

where all the coefficients of $Q_{mk'}$ lie in the maximal ideal of R . For all $k' < k$, we can write

$$A_{mk'}^{(j)} = v^{i+1} P_{mk'} + Q_{mk'}$$

where all the coefficients of $Q_{mk'}$ lie in the maximal ideal of R .

In particular, if $k' \geq k$, then $d_R(A_{mk'}^{(j)}) \geq i$ whereas if $k' < k$ then $d_R(A_{mk'}^{(j)}) \geq i + 1$.

Proof. Modulo m_R , the matrices $\overline{A}^{(j)}$ are of the form $\tilde{w}_j \mathcal{I}$, and thus every entry to the left (in the same row) of a pivot of degree i is divisible by v^{i+1} , while every entry to the right is divisible by v^i . \square

The following key Proposition shows that by suitable row operations via the elementary matrices $U_{ik}(x^{(j)})$, $L_{ik}(x^{(j)})$, $D_{ii}(x)$, we can strictly increase the defect of an entry of $A^{(j)}$ *without decreasing the total defect of $A^{(j)}$* .

Proposition 4.9. *Keep the notations and assumptions of Lemma 4.8. Assume that (m, k) is a pivot of $A^{(j)}$. There exists $x \in R[[v]]$ such that, by letting*

$$A'^{(j)} \stackrel{\text{def}}{=} \begin{cases} U_{m'm}(x)A^{(j)} & \text{if } m' > m, \\ D_{mm}(x)A^{(j)} & \text{if } m' = m, \\ L_{m'm}(x)A^{(j)} & \text{if } m' < m, \end{cases}$$

one has $\delta(A'^{(j)})_{m'k} > \delta(A^{(j)})_{m'k}$, $\delta(A'^{(j)})_{rs} \geq \min(\delta(A^{(j)})_{rs}, \delta(A^{(j)}) + 1)$ unless $r = m', s > k$, and $\delta(A'^{(j)}) \geq \delta(A^{(j)})$.

Proof. Let us write $A^{(j)} = B_{\tilde{w}_j}^{(j)} + E^{(j)}$ as in definition 4.4 and let $\delta = \delta(A^{(j)})$ be the total defect of $A^{(j)}$. Let $i \in \{0, 1, 2\}$ be the degree of the pivot of $A^{(j)}$ at (m, k) . As $\overline{A}_{mk}^{(j)} \in \overline{R}[[v]]$ is a monomial in v (cf. Definition 4.6), we can write $A_{mk}^{(j)} = u_{mk}v^i + Q_{mk}$ for some unit $u_{mk} \in R^\times$ and some element $Q_{mk} \in R[[v]]$ verifying $d_R(Q_{mk}) \geq 3$.

Let us consider the case $m' > m$. By the definition of the error term $E^{(j)}$, we have $E_{m'k}^{(j)} \in v^i R[[v]]$. In particular, we can write $E_{m'k}^{(j)} = v^i P_{m'k}$ for some $P_{m'k} \in R[[v]]$ verifying $d_R(P_{m'k}) = d_R(E_{m'k}^{(j)}) - i \geq \delta - i$. We set $x \stackrel{\text{def}}{=} -u_{mk}^{-1}P_{m'k}$. By writing $A'^{(j)} = B_{\tilde{w}_j}^{(j)} + E'^{(j)}$ as in Definition 4.4, we see that $E_{m'k}^{(j)} = xQ_{mk}$. By the above and Proposition 4.3, we have

$$\begin{aligned} d_R(E_{m'k}^{(j)}) &\geq d_R(Q_{mk}) + d_R(P_{m'k}) \\ &\geq 3 + \delta - i > \delta \end{aligned}$$

We now verify that $\delta(A'^{(j)}) \geq \delta$. Indeed, we have $A'_{ik'}^{(j)} = A_{ik'}^{(j)}$ for all $i \neq m'$ and $1 \leq k' \leq 3$.

If $i = m'$ and $k' \neq k$ we have

$$A'_{m'k'}^{(j)} = A_{m'k'}^{(j)} - u_{mk}^{-1}P_{m'k}A_{mk'}^{(j)}$$

and by Lemma 4.8, we conclude that $d_R(P_{m'k}A_{mk'}^{(j)}) \geq i + d_R(P_{m'k}) \geq \delta$, and that the inequality is strict unless $k' > k$. This completes the proof in the case $m' > m$. The other cases are similar. \square

Remark 4.10. The x used in the proof of Proposition 4.9 always has the property that $d_R(x) \geq d(A^{(j)}) - 2$ since $i \leq 2$. Thus, it will always satisfy the hypotheses of Proposition 4.5. In fact, it will continue to satisfy the hypotheses even after we increase the defect of $A^{(j)}$ by 1.

Proposition 4.11. *Let $\mathfrak{M} \in Y^{[0,2],\tau}(R)$ and let $\overline{\beta}$ be a gauge basis of $\overline{\mathfrak{M}}$. Let β be an eigenbasis of \mathfrak{M} lifting $\overline{\beta}$ and for all $0 \leq j \leq f-1$ set $A^{(j)} = \text{Mat}_\beta(\phi_{\mathfrak{M},s_{j+1}(3)}^{(j)})$. There exists another eigenbasis β' lifting $\overline{\beta}$ such that*

$$d(A'^{(j)}) > d(A^{(j)})$$

for all $0 \leq j \leq f-1$, where $A^{(j)} \stackrel{\text{def}}{=} \text{Mat}_{\beta'}(\phi_{\mathfrak{M}, s_{j+1}(3)}^{(j)})$. Furthermore, if

$$\text{Ad}_{s_j}^{-1}(u^{\mathbf{a}_1}, u^{\mathbf{a}_2}, u^{\mathbf{a}_3}) \left(D^{(j)} \right) = I^{(j)}$$

as in Proposition 2.15, then $d_R(I^{(j)}) \geq d_R(A^{(j)}) - 2$

Proof. Let $(1, k_1)$, $(2, k_2)$ and $(3, k_3)$ be the pivot entries for $A^{(j)}$, and put $\delta = d(A^{(j)})$. Using the operations as in Proposition 4.9 for the pivot $(1, k_1)$ and Proposition 4.5, we can find a change of basis such that with respect to the new basis, the matrix $A^{(j)}$ will have entries in its first column of defect $> \delta$. Apply the same argument for the second and the third column, we arrive at the desired new basis, noting that by performing the elementary operations in this order, Proposition 4.9 guarantees that we do not lose the increased defect of an entry that was already made to have defect $> \delta$. \square

Lemma 4.12. *Let $(x_\ell)_{\ell \geq 1}$ be elements of $R[[v]]$. If $\lim_{\ell \rightarrow \infty} d_R(x_\ell) = \infty$, then there exists $x \in R[[v]]$ such that $x = \sum_{\ell=1}^{\infty} x_\ell$.*

Proof. This is because $R[[v]]$ is (m_R, v) -adically complete. \square

Proof of Theorem 4.1: By a repeated application of Proposition 4.11, we can find a sequence of bases whose change of basis matrix converge by the above Lemma. Taking the limit change of basis matrix produces an eigenbasis with respect to which $A^{(j)}$ has the desired form. \square

4.2. Height conditions. Let $\overline{\mathfrak{M}} \in Y_{\mathbf{w}}^{\mu, \tau}(\mathbb{F})$. We now compute the universal lift of $\overline{\mathfrak{M}}$ with height conditions. Fix a gauge basis $\overline{\beta}$ mod p of $\overline{\mathfrak{M}}$ (Definition 2.23).

Definition 4.13. Let R be a complete local Noetherian \mathcal{O} -algebra and let $\mathfrak{M} \in Y^{[0,2], \tau}(R)$ lifting $\overline{\mathfrak{M}}$. An eigenbasis β lifting $\overline{\beta}$ is called a *gauge basis* if the matrix $\tilde{A}_{\tilde{w}_j}^{(j)} \stackrel{\text{def}}{=} \text{Mat}_{\beta}(\phi_{\mathfrak{M}, s_{j+1}(3)}^{(j)})$ satisfies the degree conditions in the third column, row \tilde{w}_j in the table 4.

We now consider the problem of deforming $(\overline{\mathfrak{M}}, \overline{\beta})$. For any Artinian \mathcal{O} -algebra A with residue field \mathbb{F} , let $D_{\overline{\mathfrak{M}}}^{\tau, \overline{\beta}}(A)$ be the category of pairs $(\mathfrak{M}_A, \beta_A)$ deforming $(\overline{\mathfrak{M}}, \overline{\beta})$ where $\mathfrak{M}_A \in Y^{\mu, \tau}(A)$ and β_A is a gauge basis of \mathfrak{M}_A . This is representable by a complete local Noetherian \mathcal{O} -algebra $R_{\overline{\mathfrak{M}}}^{\tau, \overline{\beta}}$. Explicitly, we can write down the universal matrix $\tilde{A}_{\tilde{w}_j}^{(j)}$ lifting $\text{Mat}_{\overline{\beta}}(\phi_{\overline{\mathfrak{M}}, s_{j+1}(3)}^{(j)})$ for each j satisfying the degree conditions from Table 4. $R_{\overline{\mathfrak{M}}}^{\tau, \overline{\beta}}$ is the p -flat quotient obtained by imposing height and determinant conditions.

The key point is that the bounds on the coefficients in $\tilde{A}_{\tilde{w}_j}^{(j)}$ are restrictive enough that the constraints imposed by the finite height conditions can be easily and explicitly computed. This

allows us to determine a minimal set of equations as local coordinates for $Y^{\mu,\tau}$ at $\overline{\mathfrak{M}}$. Note that $R_{\overline{\mathfrak{M}}}^{\tau,\overline{\beta}}$ has dimension $7f$ over \mathcal{O} (the monodromy condition will be computed in section 5).

Theorem 4.14. *Let $(\mathfrak{M}^{\text{univ}}, \beta^{\text{univ}})$ be the universal family over $R_{\overline{\mathfrak{M}}}^{\tau,\overline{\beta}}$. Then $\text{Mat}_{\beta^{\text{univ}}}(\phi_{\mathfrak{M}^{\text{univ}}, s_{j+1}(3)}^{(j)})$ is given in column 4 of Table 4. Furthermore,*

$$R_{\overline{\mathfrak{M}}}^{\tau,\overline{\beta}} \cong \widehat{\otimes}_j R_{\tilde{w}_j}^{\text{expl}}$$

for $R_{\tilde{w}_j}^{\text{expl}}$ given in Table 5.

The p -adic Hodge type $(2, 1, 0)$ condition is imposed by flat closure from the generic fiber. Theorem 5.13 and Corollary 5.12 in [CL] give a characterization of points of $Y^{\tau,\mu}$ for p -flat and reduced \mathcal{O} -algebras R . In this setting, this translates into the following:

Proposition 4.15. *Let R be a complete local Noetherian flat reduced \mathcal{O} -algebra. Consider $\mathfrak{M}_R \in Y^{[0,h],\tau}(R)$ for some h and let $A^{(j)} = \text{Mat}_{\beta}(\phi_{\mathfrak{M}, s_{j+1}(3)}^{(j)})$ for any eigenbasis β of \mathfrak{M}_R . Then $\mathfrak{M}_R \in Y^{\mu,\tau}(R)$ if and only if*

- (1) $\det(A^{(j)}) = x_j^* P(v)^3$ for $x_j^* \in R[[v]]^\times$;
- (2) $P(v)^2 (A^{(j)})^{-1} \in \text{Mat}_3(R[[v]][1/p])$

We can make 4.15 even more concrete. Letting $\tilde{A}_{\tilde{w}_j}^{(j)}$ be the universal matrix lifting $\overline{A}^{(j)}$ which satisfies the degree conditions in third column, row \tilde{w}_j of Table 4, then $R_{\overline{\mathfrak{M}}}^{\tau,\overline{\beta}}$ is the \mathcal{O} -flat quotient obtained by imposing the conditions

- i) for all $1 \leq i, k \leq 3$ the (ik) -minor satisfies $(\tilde{A}_{\tilde{w}_j}^{(j)})^{(ik)} \equiv 0$ modulo $P(v)$;
- ii) $\det(\tilde{A}_{\tilde{w}_j}^{(j)}) = x_j^* P(v)^3$.

The effect of conditions i), ii) are recorded in column 4 of Table 4. This is a case-wise computation; we include the computations for a few shapes. The rest are similar.

4.2.1. *The $\alpha\beta\alpha$ cell.* Assume that $\overline{\mathfrak{M}}^{(j)}$ has shape $\tilde{w}_j = \alpha\beta\alpha$. From Theorem 4.1, we deduce that

$$\tilde{A}_{\alpha\beta\alpha}^{(j)} = \begin{pmatrix} c_{11} & c_{12} & c_{13} + P(v)c_{13}^* \\ 0 & \tilde{c}_{22} + P(v)c_{22}^* & \tilde{c}_{23} + P(v)c_{23} \\ c_{31}^* v & c_{32} v & c_{33} + P(v)c_{33}' \end{pmatrix}$$

where $c_{13}^*, c_{31}^*, c_{22}^*$ are units.

Let us consider first condition i). Using p -flatness, the congruence $(\tilde{A}_{\alpha\beta\alpha}^{(j)})^{(13)} \equiv 0$ implies $\tilde{c}_{22} = 0$ and, similarly, $(\tilde{A}_{\alpha\beta\alpha}^{(j)})^{(12)} \equiv 0$ implies $\tilde{c}_{23} = 0$. This implies in particular that $(\tilde{A}_{\alpha\beta\alpha}^{(j)})^{(11)} \equiv 0$ and $(\tilde{A}_{\alpha\beta\alpha}^{(j)})^{(3k)} \equiv 0$ for all $k = 1, 2, 3$.

Again, we deduce from p -flatness and $\left(\tilde{A}_{\alpha\beta\alpha}^{(j)}\right)^{(2k)} \equiv 0$ that

$$c_{12}c_{33} = -pc_{32}c_{13}, \quad c_{11}c_{33} = -pc_{31}^*c_{13}, \quad c_{11}c_{32} = c_{31}^*c_{12}$$

(for $k = 1, 2, 3$ respectively).

Granting condition i), we therefore deduce that condition ii) is equivalent to

$$(c_{11}c_{33} + pc_{31}^*c_{13}) + P(v)(c'_{33}c_{11} - c_{31}^*c_{13} + pc_{31}^*c_{13}^*) - c_{31}^*c_{13}^*P(v)^2 = x^*P(v)^2$$

which implies that

$$E(v)(c'_{33}c_{11} - c_{31}^*c_{13} + pc_{31}^*c_{13}^*) - c_{31}^*c_{13}^*P(v)^2 = x^*P(v)^2.$$

We conclude that the condition i) and ii) hold if and only if the following equations are satisfied

$$\begin{aligned} c_{12}c_{33} &= -pc_{32}c_{13}, & c_{11}c_{33} &= -pc_{31}^*c_{13}, & c_{11}c_{32} &= c_{31}^*c_{12} \\ c'_{33}c_{11} - c_{31}^*c_{13} + pc_{31}^*c_{13}^* &= 0 \end{aligned}$$

which is exactly the condition appearing in the fourth column, $\alpha\beta\alpha$ -row in the table 4.

4.2.2. *The $\beta\alpha$ cell.* Assume that $\tilde{w}_j = \beta\alpha$. From Theorem 4.1 we can write

$$\tilde{A}_{\beta\alpha}^{(j)} = \begin{pmatrix} c_{11} & c_{12} + P(v)c_{12}^* & c_{13} \\ 0 & c_{22} + P(v)c'_{22} & c_{23} + P(v)c_{23}^* \\ vc_{31}^* & vc_{32} & c_{33} + P(v)c'_{33} \end{pmatrix}$$

where $c_{12}^*, c_{23}^*, c_{31}^*$ are units.

We consider first condition i). From $\left(\tilde{A}_{\alpha\beta\alpha}^{(j)}\right)^{(12)} \equiv 0$ and $\left(\tilde{A}_{\alpha\beta\alpha}^{(j)}\right)^{(13)} \equiv 0$, we deduce $c_{23} = 0$ and $c_{22} = 0$ respectively.

In particular, $\left(\tilde{A}_{\alpha\beta\alpha}^{(j)}\right)^{(11)} \equiv 0$ and $\left(\tilde{A}_{\alpha\beta\alpha}^{(j)}\right)^{(3k)} \equiv 0$ are automatically satisfied for all $k = 1, 2, 3$. From $\left(\tilde{A}_{\alpha\beta\alpha}^{(j)}\right)^{(2k)} \equiv 0$, we deduce, for $k = 2, 3$ respectively,

$$(4.1) \quad c_{11}c_{33} = -pc_{31}^*c_{13}, \quad c_{11}c_{32} = c_{31}^*c_{12}$$

and from the p -flatness of R we conclude that $\left(\tilde{A}_{\alpha\beta\alpha}^{(j)}\right)^{(21)} \equiv 0$.

As for the determinant condition, we obtain granting i):

$$\begin{aligned} &P(v)(c'_{22}(c_{11}c_{33} + pc_{31}^*c_{13}) + pc_{23}^*(c_{32}c_{11} - c_{31}^*c_{12})) + \\ &P(v)^2(c_{11}c'_{22}c'_{33} + c_{12}c_{23}^*c_{31}^* - pc_{31}^*c_{12}^*c_{23}^* - c_{31}^*c'_{22}c_{13} - c_{11}c_{32}c_{23}^*) + \\ &c_{12}^*c_{23}^*c_{31}^*P(v)^3 = x^*P(v)^3 \end{aligned}$$

which gives the equation

$$(4.2) \quad c_{11}c'_{22}c'_{33} + c_{12}c_{23}^*c_{31}^* - pc_{31}^*c_{12}^*c_{23}^* - c_{31}^*c'_{22}c_{13} - c_{11}c_{32}c_{23}^* = 0.$$

As $c_{11}c_{32}c_{23}^* = c_{31}^*c_{12}c_{23}^*$, the equations 4.1, 4.2 yield precisely the conditions appearing in the fourth column, $\beta\alpha$ -row in the table 4.

The computations to determine the finite height and determinant equations for the other cells are strictly analogous and left to the reader.

4.3. Gauge basis. We now consider the question of the uniqueness of the gauge basis constructed by the algorithm from the previous section. While the basis is not unique, it is essentially unique up to component-wise scaling by a torus. Let $\mathfrak{M} \in Y_{\overline{\mathfrak{M}}}^{[0,2],\tau}(R)$. Any eigenbasis β for \mathfrak{M} induces an eigenbasis on $\mathfrak{M}/u\mathfrak{M}$ (i.e., a basis for $\mathfrak{M}^{(j)}/u\mathfrak{M}^{(j)}$ for each j compatible with the linear action of descent datum). We denote this by $\beta \bmod u$.

Theorem 4.16. *Let $\mathfrak{M}, \overline{\mathfrak{M}}, \overline{\beta}$ be as in Definition 4.13. The map*

$$\beta \mapsto \beta \bmod u$$

induces a bijection between gauge bases of \mathfrak{M} and eigenbases of $\mathfrak{M}/u\mathfrak{M}$ lifting $\overline{\beta} \bmod u$.

The key consequence of Theorem 4.16 which we will use in the next section is that the addition of a gauge basis is a formally smooth operation.

Proof. Given a gauge basis $\beta = (\beta^{(j)})$, scaling any $\beta^{(j)}$ by the diagonal torus $T(R)$ gives a new gauge basis. Hence, the map is surjective.

It suffices then to show that if β_1 and β_2 are two gauge bases such that

$$(4.3) \quad \beta_1 \bmod u = \beta_2 \bmod u$$

then $\beta_1 = \beta_2$.

Let us write $\tilde{A}_i^{(j)} \stackrel{\text{def}}{=} \text{Mat}_{\beta_i}(\phi_{\mathfrak{M}, s_{j+1}(3)})$ for $i = \{1, 2\}$ (we omit the subscript \tilde{w}_j to ease notation). Then the change of basis formula (2.4) gives us

$$\tilde{A}_2^{(j)} s_{j+1}^{-1} s_j (\text{Ad}(v^{a_{s_j(1), f-j-1}}, v^{a_{s_j(2), f-j-1}}, v^{a_{s_j(3), f-j-1}}) \cdot \varphi(\text{id}_3 + I^{(j)})) s_j^{-1} s_{j+1} = (\text{id}_3 + I^{(j+1)}) \tilde{A}_1^{(j)}$$

where $v|I^{(j)}, v|I^{(j+1)}$. By the genericity assumption we see, as in the proof of Lemma 4.5 that

$$s_{j+1}^{-1} s_j (\text{Ad}(v^{a_{s_j(1), f-j-1}}, v^{a_{s_j(2), f-j-1}}, v^{a_{s_j(3), f-j-1}}) \cdot \varphi(I^{(j)})) s_j^{-1} s_{j+1} = v^3 M^{(j)}$$

where $M^{(j)} \in \text{Mat}_3(R[[v]])$ verifies $d_R(M^{(j)}) \geq d_R(I^{(j)})$. On the other hand we obtain:

$$(4.4) \quad \tilde{A}_2^{(j)} + v^3 \tilde{A}_2^{(j)} M^{(j)} = \tilde{A}_1^{(j)} + I^{(j+1)} \tilde{A}_1^{(j)}.$$

From equation (4.4), we now show that for all $n \in \mathbb{N}$, $d_R(I^{(j)}) \geq n$ for all $j = 0, \dots, f-1$, i.e., that $I^{(j)} = 0$, for all $j = 0, \dots, f-1$. Suppose we have $d_R(I^{(j)}) \geq \delta$ for all j .

Set $\overline{A}^{(j)} \stackrel{\text{def}}{=} \tilde{A}_1^{(j)} \otimes_R \mathbb{F} = \tilde{A}_2^{(j)} \otimes_R \mathbb{F}$. We define a pivot $(k(1), m(1)) \in \{1, 2, 3\}^2$ of degree $i(1)$ (cf. Definition 4.6) via the requirement that $\overline{A}_{km(1)} = 0$ for all $k \neq k(1)$ and $i(1)$ is minimal among the degrees of the pivots of $\overline{A}^{(j)}$. Similarly, we define a pivot $(k(2), m(2)) \in \{1, 2, 3\}^2$ of degree $i(2)$ via the requirement that $\overline{A}_{km(2)} = 0$ for all $k \neq k(1), k(2)$ and $i(2)$ is minimal among the degrees of the pivots of $\overline{A}^{(j)}$ which are different from $(k(1), m(1))$. We write $(k(3), m(3))$ for the remaining pivot, of degree $i(3)$. Table 3 shows that a choice of pivots like this exists, because each $\overline{A}^{(j)}$ is obtained from an upper triangular matrix by permuting rows and columns. Note that $(i(1), i(2), i(3)) = (0, 1, 2)$ or $(1, 1, 1)$.

For instance, in shape $\alpha\beta\alpha$, we have $(k(1), m(1)) = (3, 1)$, $(k(2), m(2)) = (2, 2)$ and $(k(3), m(3)) = (1, 3)$ and they all have degree 1.

For $l \in \{1, 2\}$, we have $d_R((\tilde{A}_1^{(j)})_{km(l)}) \geq i(l)$ for all $k \in \{1, 2, 3\}$. Furthermore, $d_R((\tilde{A}_1^{(j)})_{km(1)}) \geq 3$ if $k \neq k(1)$ and $d_R((\tilde{A}_1^{(j)})_{km(2)}) \geq 3$ for $k \neq k(1), k(2)$.

If $i(3) = 1$, then one still has $d_R((\tilde{A}_1^{(j)})_{km(3)}) \geq i(3)$ for all k but, when $i(3) = 2$ then one loses precision and we just have $d_R((\tilde{A}_1^{(j)})_{km(3)}) + 1 \geq i(3)$ for $k \neq k(3)$. Moreover, since a pivot reduces to a monomial modulo the maximal ideal of R , we have $(\tilde{A}_1^{(j)})_{k(l)m(l)} = x_l^* v^{i(l)} + E_l$ where $d_R(E_l) \geq 3$ and $x_l^* \in R^\times$.

For all $n \in \{1, 2, 3\}$, comparing the $nm(1)$ -th entry of equation 4.4, and truncating above degree $i(1)$ one has

$$I_{nk(1)}^{(j+1)} x_l^* v^{i(l)} + I_{nk(1)}^{(j+1)} E_l + \text{Tr}_{i(1)}(I_{nk(2)}^{(j+1)} (\tilde{A}_1^{(j)})_{k(2)m(1)}) + \text{Tr}_{i(1)}(I_{nk(3)}^{(j+1)} (\tilde{A}_1^{(j)})_{k(3)m(1)}) = v^3 (\tilde{A}_2^{P(j)} M^{(j)})_{nm(1)}$$

Here we use that the truncation kills off the contribution of $\tilde{A}_2^{(j)} - \tilde{A}_1^{(j)}$, and $v|I^{(j+1)}$. Since every term in the equation except the leftmost term has $d_R \geq \delta + 3$, we conclude that $d_R(I_{nk(1)}^{(j+1)}) \geq \delta + 2$.

Similarly, by comparing the $nk(2)$ entries and truncating, using that $d_R(I_{nk(1)}^{(j+1)}) \geq \delta + 2$, we also have $d_R(I_{nk(2)}^{(j+1)}) \geq \delta + 2$. Finally comparing the $nk(2)$ entries and truncating, and using $d_R(I_{nk(l)}^{(j+1)}) \geq \delta + 2$ for $l = 1, 2$, we get $d_R(I_{nk(3)}^{(j+1)}) \geq \delta + 1$ (here the loss of -1 is because of the weaker estimate $d_R((\tilde{A}_1^{(j)})_{km(3)}) + 1 \geq i(3)$).

□

5. MONODROMY AND POTENTIALLY CRYSTALLINE DEFORMATION RINGS

In the previous section, we essentially computed the finite height G_{K_∞} Galois deformation rings. We will now describe (framed) potentially crystalline deformation rings $R_{\bar{\rho}}^{(2,1,0),\tau}$ of p -adic Hodge type $(2,1,0)$ at each embedding and Galois type τ . The dimension of $\mathrm{Spec} R_{\bar{\rho}}^{(2,1,0),\tau}[1/p]$ is one less than that of the finite height G_{K_∞} -deformation space, the difference being the existence of a monodromy operator (cf. [Kis06]). We describe this condition explicitly in Theorem 5.6. In most cases, it can be described by a single equation on the generic fiber. Although the equations involve power-series, it can be expressed as a polynomial condition plus a transcendental part which is divisible by a power of p (due to the genericity condition).

In section §5.3, we obtain integral equations for the deformation rings by analyzing the p -flatness properties of these equations. As a result, we obtain descriptions of the special fibers of the deformation spaces. In §7, we use these descriptions to prove instances of the Serre weight conjectures and modularity lifting.

5.1. Monodromy condition. We begin by recalling some notations from [Kis06]. Let $\mathcal{O}^{\mathrm{rig}}$ denote the ring of rigid analytic functions on the open unit disc over K . We fix an embedding $\mathcal{O}^{\mathrm{rig}} \hookrightarrow K[[u]]$, i.e. identify $\mathcal{O}^{\mathrm{rig}}$ with the ring of power series $\sum_{i=0}^{\infty} a_i u^i$ where $a_i \in K$ verify $|a_i|_p r^i \rightarrow 0$ for all $r < 1$ (and hence $\mathfrak{S}[1/p]$ is identified with the subring of bounded functions on the open unit disc). Set

$$\lambda = \prod_{n=0}^{\infty} \varphi^n \left(\frac{E(u)}{p} \right) \in \mathcal{O}^{\mathrm{rig}}.$$

We define a derivation on $\mathcal{O}^{\mathrm{rig}}$ by $N_{\nabla} \stackrel{\mathrm{def}}{=} -u\lambda \frac{d}{du}$; the Frobenius on \mathfrak{S} extends to a Frobenius φ on $\mathcal{O}^{\mathrm{rig}}$. If Λ is a finite flat \mathcal{O}_E -algebra, we define $\mathcal{O}_{\Lambda}^{\mathrm{rig}} \stackrel{\mathrm{def}}{=} \mathcal{O}^{\mathrm{rig}} \otimes_{\mathbb{Z}_p} \Lambda$. For any Kisin module $\mathfrak{M}_{\Lambda} \in Y^{[0,2],\tau}(\Lambda)$ we can define its base change to $\mathcal{O}^{\mathrm{rig}}$ as $\mathfrak{M}_{\Lambda}^{\mathrm{rig}} \stackrel{\mathrm{def}}{=} \mathfrak{M}_{\Lambda} \otimes_{\mathfrak{S}} \mathcal{O}^{\mathrm{rig}}$. We have a decomposition $\mathfrak{M}_{\Lambda}^{\mathrm{rig}} = \bigoplus_{j=0}^{f-1} \mathfrak{M}_{\Lambda}^{\mathrm{rig},(j)}$.

One has the following important result:

Theorem 5.1. *The module $\mathfrak{M}_{\Lambda}^{\mathrm{rig}}[1/\lambda]$ is equipped with a canonical derivation $N_{\mathfrak{M}_{\Lambda}^{\mathrm{rig}}}$ over N_{∇} such that*

$$(5.1) \quad N_{\mathfrak{M}_{\Lambda}^{\mathrm{rig}}} \phi_{\mathfrak{M}_{\Lambda}^{\mathrm{rig}}} = E(u) \phi_{\mathfrak{M}_{\Lambda}^{\mathrm{rig}}} N_{\mathfrak{M}_{\Lambda}^{\mathrm{rig}}}$$

and $N_{\mathfrak{M}_{\Lambda}^{\mathrm{rig}}} \bmod u = 0$. The module $\mathfrak{M}_{\Lambda}^{\mathrm{rig}}$ is stable under $N_{\mathfrak{M}_{\Lambda}^{\mathrm{rig}}}$ if and only if $T_{dd}^*(\mathfrak{M}_{\Lambda})[1/p]$ is the restriction to G_{K_∞} of a potentially crystalline representation of G_K which becomes crystalline when restricted to G_L .

Proof. This is essentially [Kis06, Corollary 1.3.15]. It is stated there without tame descent, however, using the full faithfulness of the restriction from crystalline G_L -representations to G_{L_∞} -representations (Corollary 2.1.14 loc. cit.) one can extend the result to the potentially crystalline case. \square

We remark that the monodromy operator $N_{\mathfrak{M}_\Lambda^{\text{rig}}}$ respects the decomposition $\mathfrak{M}_\Lambda^{\text{rig}} = \bigoplus_{j=0}^{f-1} \mathfrak{M}_\Lambda^{\text{rig},(j)}$. In particular, one has $N_{\mathfrak{M}_\Lambda^{\text{rig}}}^{(j+1)} \phi_{\mathfrak{M}_\Lambda^{\text{rig}}}^{(j)} = E(u) \phi_{\mathfrak{M}_\Lambda^{\text{rig}}}^{(j)} N_{\mathfrak{M}_\Lambda^{\text{rig}}}^{(j)}$ where $N_{\mathfrak{M}_\Lambda^{\text{rig}}}^{(j)}$ is the monodromy operator induced by $N_{\mathfrak{M}_\Lambda^{\text{rig}}}$ on $\mathfrak{M}_\Lambda^{\text{rig},(j)}$.

Let $\mathfrak{M}_\Lambda \in Y^{[0,2],\tau}(\Lambda)$ be as above and let $\beta = \{\beta^{(j)}\}$ be an eigenbasis for \mathfrak{M}_Λ . Given the finite height conditions on \mathfrak{M}_Λ , we always have $N_{\mathfrak{M}_\Lambda^{\text{rig}}}(\mathfrak{M}_\Lambda) \subset \frac{1}{\lambda} \mathfrak{M}_\Lambda^{\text{rig}}$ by same argument from Proposition 2.2.2 [Kis06]. In what follows, we set $C^{(j-1)} \stackrel{\text{def}}{=} \text{Mat}_\beta(\phi_{\mathfrak{M}_\Lambda}^{(j-1)})$ and define the *matrix of the monodromy at j* as $N_\infty^{(j)} \stackrel{\text{def}}{=} \text{Mat}_\beta(N_{\mathfrak{M}_\Lambda^{\text{rig}}}^{(j)})$.

We can construct $N_\infty^{(j)}$ by successive approximation as follows:

Lemma 5.2. *Let $N_0^{(j)} = 0$ for all $j \in \mathbb{Z}/f\mathbb{Z}$. For each $i \geq 1$, set*

$$N_i^{(j)} \stackrel{\text{def}}{=} E(u) C^{(j-1)} \varphi(N_{i-1}^{(j-1)}) (C^{(j-1)})^{-1} - N_\nabla(C^{(j-1)}) (C^{(j-1)})^{-1}.$$

Then $N_i^{(j)}$ converges in $\frac{1}{\lambda} \text{Mat}(\mathcal{O}_\Lambda^{\text{rig}})$ to $N_\infty^{(j)}$.

Proof. For each $j \in \mathbb{Z}/f\mathbb{Z}$, define

$$r_j = \min\{|\mathbf{a}_m^{(j)} - \mathbf{a}_k^{(j)}|, |e - (\mathbf{a}_m^{(j)} - \mathbf{a}_k^{(j)})|, 1 \leq m < k \leq 3\}.$$

By the genericity condition (2.1), one deduces easily that $r_j \geq \frac{2e}{p}$.

We show by induction that

$$(5.2) \quad \lambda(N_{i+1}^{(j)} - N_i^{(j)}) \in u^{2ep^{i-1}} \text{Mat}(\mathcal{O}_\Lambda^{\text{rig}})$$

for all j and $i \geq 1$. Thus, $\lambda N_i^{(j)}$ converges to $\lambda \tilde{N}_\infty^{(j)}$ in $\text{Mat}(\Lambda[1/p][[u]])$ which satisfies the commutation relation with Frobenius. We conclude that $\lambda \tilde{N}_\infty^{(j)} = \lambda N_\infty^{(j)} \in \text{Mat}_3(\mathcal{O}_\Lambda^{\text{rig}})$ (a priori, the convergence happens in a formal power series ring, however one can estimate the Gauss norms to see that sequence actually converges in $\mathcal{O}_\Lambda^{\text{rig}}$).

The inductive step for $i \geq 1$ follows easily from the relation

$$\lambda \left(N_{i+1}^{(j)} - N_i^{(j)} \right) = \frac{E(u)^2}{p} C^{(j-1)} \varphi \left(\lambda \left(N_i^{(j-1)} - N_{i-1}^{(j-1)} \right) \right) (C^{(j-1)})^{-1}$$

since we have $E(u)^2 (C^{(j-1)})^{-1} \in \text{Mat}(\Lambda[[u]])$ by the height condition. For the base case, we consider

$$\lambda N_1^{(j)} = \lambda N_\nabla(C^{(j-1)}) (C^{(j-1)})^{-1}.$$

By the height condition, $\lambda^2 (C^{(j-1)})^{-1} \in \text{Mat}(\mathcal{O}_\Lambda^{\text{rig}})$ so it suffices to show that

$$(5.3) \quad \frac{1}{\lambda} N_\nabla(C^{(j-1)}) \in u^{r_j} \text{Mat}(\Lambda[[u]])$$

since $r_j \geq \frac{2e}{p}$. By Proposition 2.13, we have

$$C^{(j-1)} = \text{Ad}_{s_j}(u^{\mathbf{a}_1}, u^{\mathbf{a}_2}, u^{\mathbf{a}_3})(A^{(j-1)})$$

where $(A^{(j-1)})_{mk} \in \Lambda[[u^e]]$ for all $1 \leq m, k \leq 3$ and, moreover, $(A^{(j-1)})_{mk} \in u^e \Lambda[[u^e]]$ for $1 \leq k < m \leq 3$. We conclude that $-u \frac{d}{du} C^{(j-1)}$ is divisible by u^{r_j} . \square

We now can state the following condition which controls the poles of the monodromy operator. Recall that we fixed $\pi \stackrel{\text{def}}{=} (-p)^{\frac{1}{p^J-1}}$ as a uniformizer for L .

Proposition 5.3. *Let $\mathfrak{M}_\Lambda \in Y^{[0,2],\tau}(\Lambda)$ with eigenbasis β , and write $\text{Mat}_\beta(N_{\mathfrak{M}_\Lambda^{\text{rig}}}^{(j)}) = N_\infty^{(j)} = \lim_{i \rightarrow \infty} N_i^{(j)}$ as in Lemma 5.2. Then $\mathfrak{M}_\Lambda^{\text{rig}}$ is stable under $N_{\mathfrak{M}_\Lambda^{\text{rig}}}$ if and only if*

$$\lambda N_\infty^{(j)}|_{u=\pi} = 0$$

for all j .

Proof. Since $\lambda N_\infty^{(j)} \in \text{Mat}(\mathcal{O}_\Lambda^{\text{rig}})$, $\mathfrak{M}_\Lambda^{\text{rig}}$ is stable under $N_{\mathfrak{M}_\Lambda^{\text{rig}}}$ if and only if $\lambda N_\infty^{(j)}$ is divisible by λ . Since λ has simple zeroes exactly at $\{\pi^{1/p^n}\}_{n=0}^\infty$, it suffices to show that

$$\lambda N_\infty^{(j)}|_{u=\pi^{1/p^n}} = 0$$

for all j . The commutation relation $N_{\mathfrak{M}_\Lambda^{\text{rig}}}^{(j)} \phi_{\mathfrak{M}_\Lambda^{\text{rig}}}^{(j-1)} = E(u) \phi_{\mathfrak{M}_\Lambda^{\text{rig}}}^{(j-1)} N_{\mathfrak{M}_\Lambda^{\text{rig}}}^{(j-1)}$ translates into

$$N_\infty^{(j)} C^{(j-1)} + N_\nabla(C^{(j-1)}) = E(u) C^{(j)} \varphi(N_\infty^{(j-1)}).$$

Since the Frobenius $C^{(j-1)}$ is invertible at π^{1/p^n} when $n > 0$ and $N_\nabla(C^{(j-1)})$ is divisible by λ , we see that

$$\lambda N_\infty^{(j-1)}|_{u=\pi^{1/p^{n-1}}} = 0 \implies \lambda N_\infty^{(j)}|_{u=\pi^{1/p^n}}.$$

for all $n > 0$. Thus, we are reduced to checking the pole condition at $u = \pi$. \square

By construction, $N_\infty^{(j)}$ only depends on the $C^{(j)}$. So, of course, $\lambda N_\infty^{(j)}|_{u=\pi}$ also only depends on the $C^{(j)}$. In general, however this could be a complicated condition on the coefficients of $C^{(j)}$. We now show in fact this condition can be written as an explicit polynomial equation plus an “error” term which is divisible by p^2 . The special fiber of $R_{\overline{\rho}}^{(2,1,0),\tau}$ will only depend on the “leading term.”

We will want to apply our condition to the universal finite height deformations constructed in §4.2. Let R be any complete local Noetherian flat \mathcal{O} -algebra with finite residue field. Define $\mathcal{O}_R^{\text{rig}}$ be the power series $\sum_{i=0}^\infty a_n u^n$ with $a_n \in R[1/p]$ such that $p^n a_n^k \rightarrow 0$ for all $k > 0$.

Let $\mathfrak{M}_R \in Y^{[0,2],\tau}(R)$ equipped with an eigenbasis β . As before, we let $C^{(j)} \stackrel{\text{def}}{=} \text{Mat}_\beta(\phi_{\mathfrak{M}_R}^{(j)})$ and write $\text{Mat}_\beta(N_{\mathfrak{M}_R^{\text{rig}}}^{(j)}) = N_\infty^{(j)} = \lim_{i \rightarrow \infty} N_i^{(j)}$ as in Lemma 5.2. Note that $N_\infty^{(j)} \in \frac{1}{\lambda} \text{Mat}_3(\mathcal{O}_R^{\text{rig}})$.

We will now study the convergence in Lemma 5.2 more carefully.

Lemma 5.4. *We have*

$$\frac{1}{\lambda} N_\nabla(C^{(j-1)}) = \text{Ad}_{s_j}(u^{\mathbf{a}_1}, u^{\mathbf{a}_2}, u^{\mathbf{a}_3})(A^{(j-1),\dagger})$$

where $A^{(j),\dagger} \in \text{Mat}_3(R[[v]])$. Furthermore, $A^{(j),\dagger} \bmod v$ is strictly upper triangular.

Proof. Applying Leibniz rule to (2.3), we see that

$$A^{(j-1),\dagger} = -u \frac{d}{du} A^{(j-1)} - \text{Diag}(\mathbf{a}_{s_j(1)}^{(j)}, \mathbf{a}_{s_j(2)}^{(j)}, \mathbf{a}_{s_j(3)}^{(j)}) A^{(j-1)} + A^{(j-1)} \text{Diag}(\mathbf{a}_{s_j(1)}^{(j)}, \mathbf{a}_{s_j(2)}^{(j)}, \mathbf{a}_{s_j(3)}^{(j)})$$

which is a matrix in v . Furthermore, $-u \frac{d}{du} A^{(j-1)}$ is divisible by v and the remainder is 0 along the diagonal. \square

Definition 5.5. Define the *leading term* as

$$P_N(A^{(j-1)}) \stackrel{\text{def}}{=} A^{(j-1),\dagger} P(v)^2 (A^{(j-1)})^{-1}.$$

The following theorem is the main result of this section:

Theorem 5.6. *Let $\mathfrak{M}_R \in Y^{[0,2],\tau}(R)$ equipped with an eigenbasis β with $N_\infty^{(j)} \in \frac{1}{\lambda} \text{Mat}_3(\mathcal{O}_R^{\text{rig}})$ as defined above. Then*

$$\text{Ad}_{s_j}^{-1}(u^{\mathbf{a}_1}, u^{\mathbf{a}_2}, u^{\mathbf{a}_3})(\lambda N_\infty^{(j)})|_{u=\pi} = z(P_N(A^{(j-1)})|_{u=\pi} - p^2 M_{\text{err}}^{(j)})$$

where z is a unit in $R[1/p]$, $P_N(A^{(j-1)})$ is as in Definition 5.5 and $M_{\text{err}}^{(j)} \in \text{Mat}_3(R)$.

Proof. Let's examine the sequence from Lemma 5.2 in more detail. Consider that

$$\begin{aligned} N_\infty^{(j)} &= N_1^{(j)} + \sum_{i \geq 1} (N_{i+1}^{(j)} - N_i^{(j)}) \\ &= N_1^{(j)} + \sum_{i \geq 1} C^{(j-1)} \varphi(N_i^{(j-1)} - N_{i-1}^{(j-1)}) E(u) (C^{(j-1)})^{-1} \\ &= N_1^{(j)} + \sum_{i \geq 1} \left(\prod_{k=0}^{i-1} \varphi^k(C^{(j-k-1)}) \right) \varphi^i(N_1^{(j-i)}) \left(\prod_{k=i-1}^0 \varphi^k(C^{(j-k-1),*}) \right) \end{aligned}$$

where $C^{(j),*} := E(u)(C^{(j)})^{-1}$.

From Lemma 5.4, we deduce that

$$\lambda N_1^{(j)} = -\frac{\varphi(\lambda)^2}{p^2} \text{Ad}_{s_j}(u^{\mathbf{a}_1}, u^{\mathbf{a}_2}, u^{\mathbf{a}_3})(P_N(A^{(j-1)}))$$

Let $z = -\frac{\varphi(\lambda)^2}{p^2}|_{u=\pi}$ which is in $\frac{1}{p^2}\mathcal{O}_K^\times$ since $\varphi^n(E(u)/p)$ has constant term 1. Now consider the trailing term

$$p^2\lambda \left(\prod_{k=0}^{i-1} \varphi^k(C^{(j-k-1)}) \right) \varphi^i(N_1^{(j-i)}) \left(\prod_{k=i-1}^0 \varphi^k(C^{(j-k-1),*}) \right)$$

for $i \geq 1$. Substituting $N_1^{(j-i)} = \frac{\varphi(\lambda)}{p} \left(u \frac{d}{du} C^{(j-i-1)} \right) C^{(j-i-1),*}$, we can rewrite this as

$$(5.4) \quad X_i^{(j)} := \frac{\varphi^{i+1}(\lambda)^2}{p^i} \left(\prod_{k=0}^{i-1} \varphi^k(C^{(j-k-1)}) \right) \varphi^i \left(u \frac{d}{du} C^{(j-i-1)} \right) \left(\prod_{k=i}^0 \varphi^k(E(u)C^{(j-k-1),*}) \right).$$

We would now “remove the descent datum” and write this as an expression in the $A^{(j)}$ ’s. Define

$$(5.5) \quad Z_i^{(j)} = \text{Ad}_{s_j}^{-1}(u^{\mathbf{a}_1}, u^{\mathbf{a}_2}, u^{\mathbf{a}_3}) \left(\frac{1}{\varphi^{i+1}(\lambda)^2} X_i^{(j)} \right).$$

We inductively show that $Z_i^{(j)} \in \frac{v^{2p^{i-1}}}{p^i} \text{Mat}(R[[v]])$ when $i > 1$ and $Z_1^{(j)} \in \frac{v^3}{p} \text{Mat}(R[[v]])$. This suffices to prove the Theorem by evaluating at $u = \pi$ (i.e. $v = -p$).

First observe that by compatibility with descent datum, we have

$$\text{Ad}_{s_j}^{-1}(u^{\mathbf{a}_1}, u^{\mathbf{a}_2}, u^{\mathbf{a}_3})(\varphi^k(C^{(j-k-1)})), \text{Ad}_{s_j}^{-1}(u^{\mathbf{a}_1}, u^{\mathbf{a}_2}, u^{\mathbf{a}_3})(\varphi^k(E(u)C^{(j-k-1),*})) \in \text{Mat}(R[[v]]).$$

The key divisibility comes from the middle term. Take $\ell = j - i$, then

$$\begin{aligned} Y_i^{(j),\dagger} &:= \text{Ad}_{s_j}^{-1}(u^{\mathbf{a}_1}, u^{\mathbf{a}_2}, u^{\mathbf{a}_3}) \left(\varphi^i \left(u \frac{d}{du} C^{(\ell-1)} \right) \right) \\ &= s \text{Ad} \left(u^{p^i a_{s_\ell(1)}^{(\ell)} - a_{s_\ell(1)}^{(j)}}, u^{p^i a_{s_\ell(2)}^{(\ell)} - a_{s_\ell(2)}^{(j)}}, u^{p^i a_{s_\ell(3)}^{(\ell)} - a_{s_\ell(3)}^{(j)}} \right) \left(\varphi^i(A^{(j),\dagger}) \right) s^{-1} \\ &= s \text{Ad}(v^{r_{i,1}}, v^{r_{i,2}}, v^{r_{i,3}}) \left(\varphi^i(A^{(j),\dagger}) \right) s^{-1} \end{aligned}$$

for some $s \in S_3$, by the same calculation as in (2.7). We have $u^{p^i a_{s_\ell(k)}^{(\ell)} - a_{s_\ell(k)}^{(j)}} = u^{er_{i,k}} = v^{r_{i,k}}$ for some $r_{i,k}$ where $r_{i,1} > r_{i,2} > r_{i,3}$. When $i = 1$, we have $r_{1,k} = a_{s_\ell(k), f-\ell-1}$ and by the genericity condition,

$$p - 4 \geq |r_{1,1} - r_{1,2}|, |r_{1,2} - r_{1,3}| \geq 3$$

which implies that $v^3 \mid Y_1^{(j),\dagger}$. When $i > 1$, a tedious but straightforward calculation shows that

$$(p-3)p^{i-1} \geq |r_{1,1} - r_{1,2}|, |r_{1,2} - r_{1,3}| \geq 2p^{i-1}$$

and so $v^{2p^{i-1}} \mid Y_i^{(j),\dagger}$.

□

Let $\overline{\mathfrak{M}} \in Y^{\mu,\tau}(\mathbb{F})$ with shape $\mathbf{w} = (\tilde{w}_0, \tilde{w}_1, \dots, \tilde{w}_{f-1})$. Fix a gauge basis $\overline{\beta}$ on $\overline{\mathfrak{M}}$. Let $R_{\overline{\mathfrak{M}}}^{\tau, \overline{\beta}} = \widehat{\otimes}_{j \in \mathbb{Z}/f\mathbb{Z}} R_{\tilde{w}_j}^{\text{expl}}$ where $R_{\tilde{w}_j}^{\text{expl}}$ is given in the first column and \tilde{w}_j row of Table 5. This represents the universal family $(\mathfrak{M}^{\text{univ}}, \beta^{\text{univ}})$ of deformations of $(\overline{\mathfrak{M}}, \overline{\beta})$ (Theorem 4.14).

Definition 5.7. Let $A^{(j-1)} = \text{Mat}_\beta(\phi_{\mathfrak{M}^{\text{univ}}, s_j(3)}^{(j-1)})$. The *monodromy condition* at j on $(\mathfrak{M}^{\text{univ}}, \beta^{\text{univ}})$ is

$$P_N(A^{(j-1)})|_{v=-p} = p^2 M_{\text{err}}^{(j)}.$$

Let $I_{\text{mon}}^{(j)} \subset R_{\mathfrak{M}}^{\tau, \bar{\beta}}$ be the ideal generated by the nine equations from the monodromy condition at j .

The following proposition allows us to reduce, in most cases, the monodromy condition to just one equation.

Proposition 5.8. *If $\tilde{w}_{j-1} \neq \text{id}$, then $I_{\text{mon}}^{(j)}[1/p]$ is principal.*

Proof. We return again to the terms $Z_i^{(j)}$ for $i \geq 1$ defined (5.5). We can write

$$Z_i^{(j)} = A^{(j-1)} Z_i'^{(j)} P(v)^2 (A^{(j-1)})^{-1}$$

for some $Z_i'^{(j)} \in \text{Mat}_3(R_{\mathfrak{M}}^{\tau, \bar{\beta}}[1/p])$. Furthermore, we have $P_N(A^{(j-1)}) = A^{(j-1), \dagger} P(v)^2 (A^{(j-1)})^{-1}$. We also claim that $P_N(A^{(j-1)}) \equiv A^{(j-1)} Y \pmod{P(v)}$. This follows from the fact that

$$-u \frac{d}{du} \left(A^{(j-1)} P(v)^2 (A^{(j-1)})^{-1} \right) \equiv 0 \pmod{P(v)}$$

and so

$$-u \frac{d}{du} (A^{(j-1)}) P(v)^2 (A^{(j-1)})^{-1} \equiv A^{(j-1)} u \frac{d}{du} (P(v)^2 (A^{(j-1)})^{-1}) \pmod{P(v)}.$$

We conclude then the monodromy condition satisfies

$$(5.6) \quad P_N(A^{(j-1)})|_{v=-p} - p^2 M_{\text{err}} = A^{(j-1)}|_{v=-p} X = X' (P(v)^2 (A^{(j-1)})^{-1})|_{v=-p}$$

for some $X, X' \in \text{Mat}_3(R_{\mathfrak{M}}^{\tau, \bar{\beta}}[1/p])$.

Each 2×2 minor of the matrices $A^{(j-1)}|_{v=-p}$ and $(P(v)^2 (A^{(j-1)})^{-1})|_{v=-p}$ is zero by the height conditions. A survey of Table 4 shows that with p inverted $A^{(j-1)}|_{v=-p}$ has at least one unit entry as long as $\tilde{w}_{j-1} \neq \text{id}$, and the same is true for $(P(v)^2 (A^{(j-1)})^{-1})|_{v=-p}$. It follows from 5.6 that in this case the ratios between the rows and columns of $P_N(A^{(j-1)})|_{v=-p} - p^2 M_{\text{err}}$ are the same as that of $A^{(j-1)}|_{v=-p}$ and $(P(v)^2 (A^{(j-1)})^{-1})|_{v=-p}$, respectively. It follows that $I_{\text{mon}}^{(j)}[1/p]$ is generated by the entry that corresponds to a unit entry of $A^{(j-1)}|_{v=-p} (P(v)^2 (A^{(j-1)})^{-1})|_{v=-p}$, hence is principal. \square

In Table 5, we list the one equation which generates $I_{\text{mon}}^{(j)}[1/p]$.

Remark 5.9. Proposition 5.8 is *false* for the case $\tilde{w}_{j-1} = \text{id}$. The reason is that the monodromy conditions only cuts out potentially crystalline representations whose Hodge-Tate weights are in $[0, 2]$, and under the determinant condition, the Hodge-Tate weights could be either $(2, 1, 0)$ or $(1, 1, 1)$. In the id shape case, the representations with Hodge-Tate weights $(1, 1, 1)$ do show up,

and one must further refine the monodromy condition to get rid of them. This will be addressed separately in §8.

5.2. Potentially crystalline deformation rings. In the previous section, we gave a condition for Kisin module with descent datum and p -adic Hodge type $(2, 1, 0)$ to come from a potentially crystalline representation (Proposition 5.3). We will now construct a candidate for the (framed) potentially crystalline Galois deformation ring.

We begin by introducing some deformation problems. Let $\bar{\rho} : G_K \rightarrow \mathrm{GL}_3(\mathbb{F})$. Recall that $R_{\bar{\rho}}^{(2,1,0),\tau}$ is the universal *framed* potentially crystalline deformation ring with p -adic Hodge type $(2, 1, 0)$. Let $D_{\bar{\rho}}^{\tau,\square} := \mathrm{Spf} R_{\bar{\rho}}^{(2,1,0),\tau}$ denote the deformation functor. *Since we will always be working in parallel weight $(2, 1, 0)$, we omit the p -adic Hodge type in the notation.*

Assume there exists $\bar{\mathfrak{M}} \in Y^{\mu,\tau}(\mathbb{F})$ such that $T_{dd}^*(\bar{\mathfrak{M}}) \cong \bar{\rho}|_{G_{K_\infty}}$. By Theorem 3.2, if such a Kisin module exists, then it is unique. Furthermore, we fix an isomorphism $\bar{\gamma} : T_{dd}^*(\bar{\mathfrak{M}}) \cong \bar{\rho}|_{G_{K_\infty}}$.

We fix a gauge basis $\bar{\beta}$ of $\bar{\mathfrak{M}}$ (in particular, $\mathrm{Mat}_\beta(\phi_{\bar{\mathfrak{M}},s_{j+1}}^{(j)}(3))$ has the form given in Table 3).

Definition 5.10. In the following definitions, all data is taken to be compatible with the corresponding data on $\bar{\mathfrak{M}}$ when reduced modulo the maximal ideal.

- (1) Let $R_{\bar{\mathfrak{M}},\bar{\rho}}^{\tau,\square}$ denote the complete local Noetherian \mathcal{O} -algebra which represents the deformation problem

$$D_{\bar{\mathfrak{M}},\bar{\rho}}^{\tau,\square}(A) := \left\{ (\mathfrak{M}_A, \rho_A, \delta_A) \mid \mathfrak{M}_A \in Y^{\mu,\tau}(A), \rho_A \in D_{\bar{\rho}}^{\tau,\square}(A), \delta_A : T_{dd}^*(\mathfrak{M}_A) \cong (\rho_A)|_{G_{K_\infty}} \right\}$$

- (2) Let $R_{\bar{\mathfrak{M}},\bar{\rho}}^{\tau,\bar{\beta},\square}$ denote the complete local Noetherian \mathcal{O} -algebra which represents the deformation problem

$$D_{\bar{\mathfrak{M}},\bar{\rho}}^{\tau,\bar{\beta},\square}(A) = \left\{ (\mathfrak{M}_A, \rho_A, \beta_A) \mid (\mathfrak{M}_A, \rho_A) \in D_{\bar{\mathfrak{M}},\bar{\rho}}^{\tau,\square}(A), \beta_A \text{ a gauge basis for } \mathfrak{M}_A \right\}$$

- (3) Let $R_{\bar{\mathfrak{M}}}^{\tau,\bar{\beta}}$ be as in Theorem 4.14 which represents the deformation problem $D_{\bar{\mathfrak{M}}}^{\tau,\bar{\beta}}(A)$.
- (4) Let $R_{\bar{\mathfrak{M}}}^{\tau,\bar{\beta},\square}$ denote the complete local Noetherian \mathcal{O} -algebra which represents the deformation problem of triples $(\mathfrak{M}_A, \beta_A, \underline{e}_A)$ where $(\mathfrak{M}_A, \beta_A) \in D_{\bar{\mathfrak{M}}}^{\tau,\bar{\beta}}(A)$ and \underline{e}_A is a basis of $T_{dd}^*(\mathfrak{M}_A)$ lifting the basis on $\bar{\rho}|_{G_{K_\infty}}$ so that $(T_{dd}^*(\mathfrak{M}_A), \underline{e}_A)$ is a framed deformation of $\bar{\rho}|_{G_{K_\infty}}$.
- (5) Let $R_{\bar{\mathfrak{M}}}^{\tau,\bar{\beta},\nabla}$ denote the \mathcal{O} -flat quotient of $R_{\bar{\mathfrak{M}}}^{\tau,\bar{\beta},\square}$ such that $\mathrm{Spec} R_{\bar{\mathfrak{M}}}^{\tau,\bar{\beta},\nabla}[1/p]$ is the vanishing locus of the monodromy equations on $\mathrm{Spec} R_{\bar{\mathfrak{M}}}^{\tau,\bar{\beta}}[1/p]$.

The relationships between the various deformation problems are summarized in the following diagram. The square is Cartesian and f.s. stands for formally smooth.

$$(5.7) \quad \begin{array}{ccccc} & & \mathrm{Spf} R_{\overline{\mathfrak{M}}}^{\tau, \overline{\beta}, \square, \nabla} & \xrightarrow{\text{f.s.}} & \mathrm{Spf} R_{\overline{\mathfrak{M}}}^{\tau, \overline{\beta}, \nabla} \\ & \nearrow \xi & \downarrow & & \downarrow \\ \mathrm{Spf} R_{\overline{\mathfrak{M}}, \overline{\rho}}^{\tau, \overline{\beta}, \square} & \hookrightarrow & \mathrm{Spf} R_{\overline{\mathfrak{M}}}^{\tau, \overline{\beta}, \square} & \xrightarrow{\text{f.s.}} & \mathrm{Spf} R_{\overline{\mathfrak{M}}}^{\tau, \overline{\beta}} \\ \downarrow \text{f.s.} & & & & \\ \mathrm{Spf} R_{\overline{\rho}}^{\mu, \tau} & \xleftarrow{\sim} & \mathrm{Spf} R_{\overline{\mathfrak{M}}, \overline{\rho}}^{\tau, \square} & & \end{array}$$

The maps which are formally smooth correspond to forgetting either a framing on the Galois representation or a gauge basis on the Kisin module. The former is clearly formally smooth while the latter is formally smooth by Theorem 4.16.

The isomorphism between $\mathrm{Spf} R_{\overline{\rho}}^{\mu, \tau}$ and $\mathrm{Spf} R_{\overline{\mathfrak{M}}, \overline{\rho}}^{\tau, \square}$ is Corollary 3.5. If $\overline{\mathfrak{M}}$ has shape $(\tilde{w}_0, \dots, \tilde{w}_{f-1})$, then $R_{\overline{\mathfrak{M}}}^{\tau, \overline{\beta}} = \widehat{\otimes} R_{\tilde{w}_j}^{\mathrm{expl}}$ where $R_{\tilde{w}_j}^{\mathrm{expl}}$ is given in Table 5 (Theorem 4.14). As we will see, as long as $\tilde{w}_j \neq \mathrm{id}$ for all j , the map ξ will be an isomorphism (see Remark 5.9)

Proposition 5.11. *Assume that $\mathrm{ad}(\overline{\rho})$ is cyclotomic free (Definition 3.7). Then the map*

$$\mathrm{Spf} R_{\overline{\mathfrak{M}}, \overline{\rho}}^{\tau, \overline{\beta}, \square} \rightarrow \mathrm{Spf} R_{\overline{\mathfrak{M}}}^{\tau, \overline{\beta}, \square}$$

between the deformation spaces defined in Definition 5.10 is a closed immersion.

Proposition 5.11 follows from the following Galois cohomology result:

Proposition 5.12. *Assume that $\mathrm{ad}(\overline{\rho})$ is cyclotomic free (Definition 3.7). Then the map*

$$Z^1(K, \mathrm{ad}(\overline{\rho})) \rightarrow Z^1(K_{\infty}, \mathrm{ad}(\overline{\rho}))$$

is injective.

Proof. If $\mathrm{ad}(\overline{\rho})$ is cyclotomic free, we can apply Lemmas 3.9 and 3.10 to $\mathrm{ad}(\overline{\rho})$ and the claim follows from the snake lemma applied to the defining sequences of Z^1 :

$$\begin{array}{ccccccc} 0 & \longrightarrow & B^1(K, \mathrm{ad}(\overline{\rho})) & \longrightarrow & Z^1(K, \mathrm{ad}(\overline{\rho})) & \longrightarrow & H^1(K, \mathrm{ad}(\overline{\rho})) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & B^1(K_{\infty}, \mathrm{ad}(\overline{\rho})) & \longrightarrow & Z^1(K_{\infty}, \mathrm{ad}(\overline{\rho})) & \longrightarrow & H^1(K_{\infty}, \mathrm{ad}(\overline{\rho})) \longrightarrow 0. \end{array}$$

□

Theorem 5.13. *Assume that $\text{ad}(\bar{\rho})$ is cyclotomic free. The map*

$$\text{Spf} R_{\overline{\mathfrak{M}}, \bar{\rho}}^{\tau, \bar{\beta}, \square} \hookrightarrow \text{Spf} R_{\overline{\mathfrak{M}}}^{\tau, \bar{\beta}, \square}$$

factors through $\text{Spf} R_{\overline{\mathfrak{M}}}^{\tau, \bar{\beta}, \square, \nabla}$ inducing a surjective map $R_{\overline{\mathfrak{M}}}^{\tau, \bar{\beta}, \square, \nabla} \xrightarrow{\xi} R_{\overline{\mathfrak{M}}, \bar{\rho}}^{\tau, \bar{\beta}, \square}$. Furthermore, if $\tilde{w}_j \neq \text{id}$ for all $j \in \mathbb{Z}/f\mathbb{Z}$ and $R_{\overline{\mathfrak{M}}}^{\tau, \bar{\beta}, \square, \nabla}$ is reduced, then ξ is an isomorphism.

Proof. Both $R_{\overline{\mathfrak{M}}}^{\tau, \bar{\beta}, \square}$ and $R_{\overline{\mathfrak{M}}, \bar{\rho}}^{\tau, \bar{\beta}, \square}$ are flat \mathcal{O} -algebras. Furthermore, $R_{\overline{\mathfrak{M}}, \bar{\rho}}^{\tau, \bar{\beta}, \square}[1/p]$ is reduced since the same is true for the potentially crystalline deformation ring $R_{\bar{\rho}}^{\mu, \tau}$. Thus, it suffices to show factorization at the level of $\overline{\mathbb{Q}}_p$ -points.

For any \mathcal{O}' finite over \mathcal{O} , an \mathcal{O}' -point of $R_{\overline{\mathfrak{M}}, \bar{\rho}}^{\tau, \bar{\beta}, \square}$ corresponds to a Kisin module $\mathfrak{M}_{\mathcal{O}'} \in Y^{\mu, \tau}(\mathcal{O}')$ such that $T_{dd}^*(\mathfrak{M}_{\mathcal{O}'})$ is a lattice in a potentially crystalline representation with Hodge-Tate weights $(2, 1, 0)$. By Theorem 5.1, the monodromy condition holds at the corresponding $\mathcal{O}'[1/p]$ point.

Any homomorphism $R_{\overline{\mathfrak{M}}}^{\tau, \bar{\beta}, \square, \nabla} \rightarrow \mathcal{O}'$ gives rise to a Kisin module $\mathfrak{M}_{\mathcal{O}'}$ together with a gauge basis on which the Frobenius has the form given in Table 4. Furthermore, $\mathfrak{M}_{\mathcal{O}'} \otimes_{\mathfrak{S}} \mathcal{O}^{\text{rig}}$ is stable under the monodromy operator and hence $T_{dd}^*(\mathfrak{M}_{\mathcal{O}'})[1/p] =: V_{E'}$ extends to a potentially crystalline representation of G_K . The claim is as long as $\tilde{w}_j \neq \text{id}$, then $V_{E'}$ has p -adic Hodge type of parallel weight $(2, 1, 0)$. The p -adic Hodge type at the embedding σ_j is $(1, 1, 1)$ if and only if the Frobenius $C^{(j)}$ is divisible by $E(u)$, equivalently $A^{(j)}$ is divisible by $P(v)$. A survey of last column of Table 4 shows that this can only happen when $\tilde{w}_j = \text{id}$. \square

Corollary 5.14. *Assume that $\text{ad}(\bar{\rho})$ is cyclotomic free. If $\tilde{w}_j \neq \text{id}$ for all $j \in \mathbb{Z}/f\mathbb{Z}$, then*

$$R_{\bar{\rho}}^{\mu, \tau} \llbracket S_1, \dots, S_{3f-1} \rrbracket \cong R_{\overline{\mathfrak{M}}, \text{red}}^{\tau, \bar{\beta}, \nabla} \llbracket T_1, \dots, T_8 \rrbracket.$$

Remark 5.15. The assumption that $\text{ad}(\bar{\rho})$ is cyclotomic free is automatic (by Proposition 3.8) if one assumes a slightly stronger genericity condition on τ which forces $\bar{\rho}$ to be 2-generic. In any case, it is likely that this assumption could be removed by using more about the tangent space of the potentially crystalline deformation ring.

5.3. Explicit deformation rings. We now proceed to compute $R_{\overline{\mathfrak{M}}}^{\tau, \bar{\beta}, \nabla}$ in many cases thus obtaining by Corollary 5.14 a description of $R_{\bar{\rho}}^{\mu, \tau}$. Assume that $\tilde{w}_j \notin \{\alpha, \text{id}\}$ for all j .

Recall that $R_{\overline{\mathfrak{M}}}^{\tau, \bar{\beta}} := \widehat{\otimes} R_{\tilde{w}_j}^{\text{expl}}$ where $R_{\tilde{w}_j}^{\text{expl}}$ is the \mathcal{O} -algebra corresponding to shape \tilde{w}_j in Table 5. Let $\tilde{I}_{\text{mon}} \subset R_{\overline{\mathfrak{M}}}^{\tau, \bar{\beta}}$ denote the p -saturation of the sum of the ideals $I_{\text{mon}}^{(j)}$ generated by the monodromy conditions (Definition 5.7). By definition,

$$R_{\overline{\mathfrak{M}}}^{\tau, \bar{\beta}, \nabla} = R_{\overline{\mathfrak{M}}}^{\tau, \bar{\beta}} / \tilde{I}_{\text{mon}}.$$

Now, we can consider the explicit quotient $R_{\overline{\mathfrak{M}}}^{\text{expl}, \nabla}$ of $R_{\overline{\mathfrak{M}}}^{\tau, \overline{\beta}}$ given by imposing the single monodromy equation for each j in Table 5. By Proposition 5.8,

$$R_{\overline{\mathfrak{M}}}^{\text{expl}, \nabla}[1/p] = R_{\overline{\mathfrak{M}}}^{\tau, \overline{\beta}} / \widetilde{I}_{\text{mon}}[1/p].$$

The aim is to determine the p -torsion free quotient of $R_{\overline{\mathfrak{M}}}^{\text{expl}, \nabla}$. In most cases, this is (non-canonically) isomorphic to a completed tensor product of rings $R_{\overline{\mathfrak{M}}, \tilde{w}_j}^{\text{expl}, \nabla}$. Table 6 gives $R_{\overline{\mathfrak{M}}, \tilde{w}_j}^{\text{expl}, \nabla}$ for each shape and depending on $\overline{\mathfrak{M}}$.

Remark 5.16. Table 6 only includes the deformation $R_{\overline{\mathfrak{M}}, \tilde{w}_j}^{\text{expl}, \nabla}$ for our chosen representatives for the δ -orbits on $\text{Adm}(2, 1, 0)$ (see Corollary 2.25 and Remark 2.26). If \tilde{w}' is in the same δ -orbit as \tilde{w} , then the explicit ring for that shape is isomorphic to $R_{\overline{\mathfrak{M}}, \tilde{w}_j}^{\text{expl}, \nabla}$, only the labelling of the variables by entry changes.

We give two sample calculations of $R_{\overline{\mathfrak{M}}, \tilde{w}_j}^{\text{expl}, \nabla}$ with the rest being similar.

5.3.1. *The $\alpha\beta\alpha$ cell.* The monodromy equation is of the form

$$c_{11}((a-b)c_{23}c_{32} - (a-c)c_{22}^*c'_{33}) = p(e-a+c)c_{31}^*c_{22}^*c_{13}^* + p^2x_{err} = pz^*$$

where z^* is a unit. Let $y'_{33} = ((a-b)c_{23}c_{32} - (a-c)c_{22}^*c'_{33})(z^*)^{-1}$ which replaces c'_{33} . We have

$$c_{11}y'_{33} = p.$$

By p -flatness, y'_{33} is not a zero-divisor, and we can multiply the first height equation by y'_{33} to get

$$y'_{33}c_{11}c_{33} = -py'_{33}c_{13}c_{31}^* \implies c_{33} = -c_{13}y''_{33}$$

thereby eliminating c_{33} . The second finite height equation can be solved to eliminate c_{13} . We are left then with the one equation

$$c_{11}y'_{33} = p.$$

There are two cases. When $(a-b)\bar{c}_{23}\bar{c}_{32} - (a-c)\bar{c}_{22}^*\bar{c}'_{33} \neq 0$, then y''_3 is unit in which case we can solve for c_{11} . Otherwise, y'_{33} is in the maximal ideal, and we are left with this one equation as in Table 6.

5.3.2. *The $\beta\alpha$ cell.* The monodromy equation is of the form

$$c_{11}((a-c)c'_{22}c'_{33} + (a-b)c_{23}^*c_{32}) = pz^*$$

where z^* is a unit. Let $y_{32} = ((a-c)c'_{22}c'_{33} + (a-b)c_{23}^*c_{32})(z^*)^{-1}$ which replaces c_{32} so that we have

$$c_{11}y_{32} = p.$$

Multiplying the first height equation by y_{32} , we get

$$y_{32}c_{11}c_{33} = -pc_{31}^*c_{13}y_{32} \implies c_{33} = -c_{31}^*c_{13}y_{32}$$

thereby eliminating c_{33} . Let $y_{13} = -(c_{11}c'_{33} - c_{13}c_{31}^*)(c_{23}^*c_{12}^*c_{31}^*)^{-1}$ which replaces c_{13} . We have reduced the equations to

$$c_{11}y_{32} = p, c'_{22}y_{13} = p.$$

There are again two cases. When $\bar{c}_{32} \neq 0$, y_{32} is a unit in which case we can solve the first equation. Otherwise, y_{32} is in the maximal ideal in which case this is a minimal set of equations.

6. BASE CHANGE

In this section, we extend the results of §4-5 to non-principal series tame types. These tame inertial types will give us more flexibility in the global applications to isolate certain combinations of Serre weights. The setup is similar to [EGS15] where deformation rings for tame cuspidal types are computed for GL_2 . The end result is that the deformation rings have essentially the same form and shapes as for the principal series types.

Let $r \in \{2, 3\}$ and define $f' \stackrel{\mathrm{def}}{=} fr$, $K' \stackrel{\mathrm{def}}{=} \mathbb{Q}_{p^{f'}}$. We write $e' \stackrel{\mathrm{def}}{=} p^{f'} - 1$, let $\pi' \stackrel{\mathrm{def}}{=} \pi^{\frac{e'}{e}}$ and set $L' \stackrel{\mathrm{def}}{=} K'(\pi')$. Recall the character $\omega_{\pi'} : I_{K'} \rightarrow W(k')^\times$. By fixing an embedding $\sigma'_0 : k' \hookrightarrow \mathbb{F}$ extending $\sigma_0 : k \hookrightarrow \mathbb{F}$, we obtain a fundamental character $\omega_{f'} \stackrel{\mathrm{def}}{=} \omega_{\pi'}^{\frac{e'}{e}} = \omega_f$.

6.1. Tame descent datum. Recall the notations and the general setting of §2.1. Consider f -tuples $\mathbf{a}_k \in \{0, \dots, p-1\}^f$ for $1 \leq k \leq 3$ together with an orientation $(s_j) \in S_3^f$.

Consider the tame (non-principal series) inertial types τ :

$$\begin{cases} \omega_{f'}^{-\mathbf{a}_1^{(0)} - p^f \mathbf{a}_1^{(0)}} \oplus \omega_{f'}^{-\mathbf{a}_2^{(0)} - p^f \mathbf{a}_3^{(0)}} \oplus \omega_{f'}^{-\mathbf{a}_3^{(0)} - p^f \mathbf{a}_2^{(0)}} & \text{when } r = 2 \\ \omega_{f'}^{-\mathbf{a}_1^{(0)} - p^f \mathbf{a}_2^{(0)} - p^{2f} \mathbf{a}_3^{(0)}} \oplus \omega_{f'}^{-\mathbf{a}_2^{(0)} - p^f \mathbf{a}_3^{(0)} - p^{2f} \mathbf{a}_1^{(0)}} \oplus \omega_{f'}^{-\mathbf{a}_3^{(0)} - p^f \mathbf{a}_1^{(0)} - p^{2f} \mathbf{a}_2^{(0)}} & \text{when } r = 3 \end{cases}$$

We write τ' for the base change of τ to K'/K (which is just τ considered as a representation of $I_{K'}$). There is a triple $(\mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3)$ with $\mathbf{a}'_k \in \{0, \dots, p-1\}^{f'}$ associated to τ' such that $\tau' = \eta_1 \oplus \eta_2 \oplus \eta_3$ with $\eta_k = \omega_f^{-\mathbf{a}'_k{}^{(0)}}$ (with the characters ordered as above). We say that the type τ is *generic* if τ' is *generic* (equivalently, the triple $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ is generic).

The following lemma records the effect of base change on the orientation:

Proposition 6.1. *Let τ be a generic tame type. For $0 \leq j \leq f-1$ and $0 \leq i \leq r-1$, define*

$$s'_{j+if} = s_\tau^{i+1} \circ s_j \in S_3$$

where $s_\tau = (2, 3)$ (resp. $s_\tau = (1, 2, 3)$) if $r = 2$ (resp. $r = 3$). Then, the f' -tuple $(s'_{j'}) \in S_3^{f'}$ is an orientation on τ' .

Proof. This is a casewise computation, remarking that the orientation at j' on $(\mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3)$ is determined, under the genericity assumption, by $(a'_{1,f'-1-j'}, a'_{2,f'-1-j'}, a'_{3,f'-1-j'})$. \square

We now study Kisin modules with descent datum of type τ in relation to Kisin modules with descent datum of type τ' . In the next subsection, we will apply this to potentially crystalline deformation ring with tame Galois type τ . We write $\sigma \in \mathrm{Gal}(K'/\mathbb{Q}_p)$ for the absolute Frobenius on K' and set $\Delta' \stackrel{\mathrm{def}}{=} \mathrm{Gal}(L'/K')$ with $L' \stackrel{\mathrm{def}}{=} K'(\pi')$.

We define a Frobenius-twist morphism

$$(\sigma^f)^* : Y^{[0,h],\tau'} \rightarrow Y^{[0,h],(\tau')^{p^f}}.$$

Let R be any \mathcal{O} -algebra and let $\mathfrak{M} \in Y^{[0,h],\tau'}(R)$. Define $(\sigma^f)^*(\mathfrak{M})$ to be the $(W(k') \otimes_{\mathbb{Z}_p} R)[[u]]$ -module obtained from \mathfrak{M} via the base change $\sigma^f : W(k') \rightarrow W(k')$. We define the Frobenius by $\phi_{(\sigma^f)^*(\mathfrak{M})} \stackrel{\text{def}}{=} (\sigma^f)^*(\phi_{\mathfrak{M}})$ and an action of Δ' via the canonical isomorphism

$$\widehat{g}^* \left((\sigma^f)^*(\mathfrak{M}) \right) \cong (\sigma^f)^* \left(\widehat{g^{p^f}}^*(\mathfrak{M}) \right).$$

using that $g \mapsto g^{p^f}$ is an automorphism of Δ' . In the Lemma below, we see that if \mathfrak{M} has type τ' , then $(\sigma^f)^*(\mathfrak{M})$ has type $(\tau')^{p^f} := (\eta'_1)^{p^f} \oplus (\eta'_2)^{p^f} \oplus (\eta'_3)^{p^f}$.

Lemma 6.2. *Let $\mathfrak{M} \in Y^{[0,h],\tau'}(R)$. For all $j \in \{0, \dots, f' - 1\}$, one has the following commutative diagram of $R[[u]]$ -modules*

$$\begin{array}{ccc} \varphi^*(((\sigma^f)^*(\mathfrak{M}))^{(j)}) & \xrightarrow{\phi_{(\sigma^f)^*(\mathfrak{M})}^{(j)}} & ((\sigma^f)^*(\mathfrak{M}))^{(j+1)} \\ \downarrow \wr & & \downarrow \wr \\ \varphi^*(\mathfrak{M}^{(j-f)}) & \xrightarrow{\phi_{\mathfrak{M}}^{(j-f)}} & \mathfrak{M}^{(j-f+1)}. \end{array}$$

Furthermore, for any character $\eta : \Delta' \rightarrow \mathcal{O}^\times$, one has the following commutative diagram of $R[[v]]$ -modules

$$\begin{array}{ccc} \varphi((\sigma^f)^*(\mathfrak{M}))_\eta^{(j)} & \longrightarrow & ((\sigma^f)^*(\mathfrak{M}))_\eta^{(j+1)} \\ \downarrow \wr & & \downarrow \wr \\ \varphi \mathfrak{M}_{\eta^{p^f}}^{(j-f)} & \longrightarrow & \mathfrak{M}_{\eta^{p^f}}^{(j-f+1)}. \end{array}$$

with horizontal maps induced by the Frobenius.

Since τ' is the base change of the tame inertial type for I_K , $(\tau')^{p^f} = \tau'$ and the Frobenius-twist induces an automorphism of $Y^{\mu,\tau'}$. We define the ‘fixed points’ of this automorphism:

Definition 6.3. For any \mathcal{O} -algebra R , define

$$Y^{\mu,\tau}(R) = \{(\mathfrak{M}, \iota) \mid \mathfrak{M} \in Y^{\mu,\tau'}(R), \iota : (\sigma^f)^*(\mathfrak{M}) \xrightarrow{\sim} \mathfrak{M}\}$$

such that $\iota \circ (\sigma^f)^* \iota = \text{id}_{\mathfrak{M}}$ (resp. $\iota \circ (\sigma^f)^* \iota \circ (\sigma^{2f})^* \iota = \text{id}_{\mathfrak{M}}$) when $r = 2$ (resp. $r = 3$).

A morphism $(\mathfrak{M}_1, \iota_1) \rightarrow (\mathfrak{M}_2, \iota_2)$ in $Y^{\mu,\tau}(R)$ is a morphism $\mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ in $Y^{\mu,\tau'}$ which commutes with the Frobenius twist.

Proposition 6.4. *Let R be an Artinian \mathcal{O} -algebra and let $(\mathfrak{M}, \iota) \in Y^{\mu, \tau}(R)$. The $G_{K'_\infty}$ -representation $T_{dd}^*(\mathfrak{M})$ admits a canonical extension to a G_{K_∞} -representation which we denote by $T_{dd'}^*(\mathfrak{M})$.*

Proof. The étale ϕ -module $\mathcal{M} := \mathfrak{M}[1/u]$ over $\mathcal{O}_{\mathcal{E}, L'}$ has an action of the group Δ' from the descent datum. The isomorphism ι extends this to action of $\text{Gal}(L'/K)$. In §2.3, we defined

$$T_{dd}^*(\mathfrak{M}) = \mathbb{V}_{dd}^*(\mathcal{M}) = \text{Hom}_{\varphi, \mathcal{O}_{\mathcal{E}, L'}}(\mathcal{M}, \mathcal{O}_{\mathcal{E}^{un}, K'})$$

Identifying $\mathcal{O}_{\mathcal{E}^{un}, K'}$ with $\mathcal{O}_{\mathcal{E}^{un}, K}$, we have a natural action of G_{K_∞} where G_{K_∞} -acts on \mathcal{M} through $\text{Gal}(L'/K)$. In this way, we construct $T_{dd'}^*$ which is a functor from $Y^{\mu, \tau}(R)$ to G_{K_∞} -representations. It is clearly compatible with T_{dd}^* . \square

Proposition 6.5. *Let $(\overline{\mathfrak{M}}, \bar{\iota}) \in Y^{\mu, \tau}(\mathbb{F})$. If $\mathbf{w} = (\tilde{w}_0, \dots, \tilde{w}_{f'-1}) \in \widetilde{W}^{f'}$ is the shape of $\overline{\mathfrak{M}}$ considered as an element of $Y^{\mu, \tau'}(\mathbb{F})$, then*

$$\tilde{w}_{j'_1} = \tilde{w}_{j'_2} \quad \text{whenever} \quad j'_1 \equiv j'_2 \pmod{f}.$$

In other words, $\overline{\mathfrak{M}}$ has parallel shape.

Proof. The isomorphism $\bar{\iota}$ induces an isomorphism $\mathfrak{M}_{\eta_{s_j(3)}}^{(j)} \cong \mathfrak{M}_{\eta_{s_j(3)}^{p^f}}^{(j-f)}$ (Lemma 6.2). The Proposition follows from the fact that $\eta_{s_j(3)}^{p^f} = \eta_{s_{j-f}(3)}$. \square

Definition 6.6. Let $\bar{\rho} : G_K \rightarrow \text{GL}_3(\mathbb{F})$ such that $T_{dd'}^*(\overline{\mathfrak{M}}) \cong \bar{\rho}|_{G_{K_\infty}}$ for some $(\overline{\mathfrak{M}}, \bar{\iota}) \in Y^{\mu, \tau}(\mathbb{F})$. Define $\mathbf{w}(\bar{\rho}, \tau) = (\tilde{w}_0, \dots, \tilde{w}_{f'-1}) \in \text{Adm}(2, 1, 0)^f$ such that \tilde{w}_j is the shape of $\overline{\mathfrak{M}} \in Y^{\mu, \tau'}(\mathbb{F})$ at j' for any $j' \equiv j \pmod{f}$. This is well-defined by Proposition 6.5.

6.2. Tame deformation rings. Let $(\overline{\mathfrak{M}}, \bar{\iota}) \in Y^{\mu, \tau}(\mathbb{F})$. Fix a gauge basis $\bar{\beta}$ on $\overline{\mathfrak{M}}$ as an element of $Y^{\mu, \tau'}$. We define the same deformation problems from Definition 5.10 but with $Y^{\mu, \tau}$ as in Definition 6.3 and using $T_{dd'}^*$ (Proposition 6.4) in place of T_{dd}^* . For instance, we now have

$$D_{\overline{\mathfrak{M}}, \bar{\rho}}^{\tau, \square}(R) \stackrel{\text{def}}{=} \left\{ (\mathfrak{M}_R, \iota, \rho_R, \delta_R) \mid (\mathfrak{M}_R, \iota) \in Y_{\overline{\mathfrak{M}}}^{\mu, \tau}(R), \rho_R \in D_{\bar{\rho}}^{\tau, \square}(R) \right. \\ \left. \text{and } \delta_R : T_{dd'}^*(\mathfrak{M}_R) \xrightarrow{\sim} \rho_R|_{G_{K_\infty}} \right\}$$

We obtain a diagram analogous to (5.7). We discuss the couple of places where the arguments differ from §4.2.

The analogue of Theorem 5.13 holds by the following Lemma:

Lemma 6.7. *Let E'/E be a finite extension. Let $V_{E'}$ be a continuous representation of G_{K_∞} . Then $V_{E'}$ extends to a potentially (for L'/K) crystalline representation of G_K if and only if $V_{E'}$ extends to a potentially (for L'/K') crystalline representation of $G_{K'}$.*

Proof. This is a consequence of the fact that the restriction from crystalline $G_{L'}$ -representations to $G_{L'_\infty}$ is fully faithful which is Corollary 2.1.14 in [Kis06]. \square

We deduce, with the same hypotheses as in Corollary 5.14, that

$$R_{\bar{\rho}}^{\mu, \tau} \llbracket S_1, \dots, S_{3f'-1} \rrbracket \cong R_{\bar{\mathfrak{M}}, \text{red}}^{\tau, \bar{\beta}, \nabla} \llbracket T_1, \dots, T_8 \rrbracket.$$

We now address the question of describing $R_{\bar{\mathfrak{M}}}^{\tau, \bar{\beta}, \nabla}$. Let $(\mathfrak{M}^{\text{univ}}, \beta^{\text{univ}})$ be the universal family over $R_{\bar{\mathfrak{M}}}^{\tau, \bar{\beta}}$. The analogue of Theorem 4.14 is the following:

Proposition 6.8. *Let \mathbf{w} be the shape of $\bar{\mathfrak{M}}$. The natural map*

$$(6.1) \quad \text{Spf} R_{\bar{\mathfrak{M}}}^{\tau, \bar{\beta}} \rightarrow \text{Spf} \widehat{\otimes}_{j \in \{0, \dots, f-1\}} R_{\tilde{w}_j}^{\text{expl}}$$

induced by $\text{Mat}_{\beta^{\text{univ}}}(\phi_{\mathfrak{M}^{\text{univ}}, s_{j+1}(3)}^{(j)})$ for $0 \leq j \leq f-1$ is formally smooth of relative dimension $3(r-1)f$.

Proof. We focus on the case when $r = 2$. The other case is similar. Consider $(\mathfrak{M}, \iota) \in Y^{\mu, \tau}(R)$ lifting $(\bar{\mathfrak{M}}, \bar{\iota})$. Let β_1 be a gauge basis for \mathfrak{M} . Then $\beta_2 := \iota((\sigma^f)^* \beta_1)$ is another gauge basis. As usual, we take

$$A_i^{(j')} \stackrel{\text{def}}{=} \text{Mat}_{\beta_i}(\phi_{\mathfrak{M}, \eta_{s_{j'}(3)}}^{(j')})$$

for all $0 \leq j' \leq f'-1$.

By Lemma 6.2 and Proposition 6.5, we have that

$$A_1^{(j')} = A_2^{(j'+f)}$$

For $f \leq j' \leq 2f-1$, we have two gauge bases $\beta_1^{(j')}$ and $\beta_2^{(j')}$ on $\mathfrak{M}^{(j')}$. By Theorem 4.16, there exists diagonal matrix $D^{(j')} \in \text{GL}_3(R)$ such that $D^{(j')} \beta_1^{(j')} = \beta_2^{(j')}$.

The data of $\{A_1^{(j')} \mid 0 \leq j' \leq f-1\}$ and $\{D^{(j')} \mid f \leq j' \leq 2f-1\}$ determines $(\mathfrak{M}, \iota, \beta_1)$ up to isomorphism. The map to $\text{Spf} \widehat{\otimes}_{j \in \{0, \dots, f-1\}} R_{\tilde{w}_j}^{\text{expl}}$ corresponds to forgetting the $D^{(j')}$, this is formally smooth. It is relative dimension $3(r-1)f$. \square

Finally, we conclude with the addition of monodromy.

Theorem 6.9. *Let $R_{\bar{\mathfrak{M}}}^{\tau, \bar{\beta}, \nabla}$ be the flat \mathcal{O} -quotient of $R_{\bar{\mathfrak{M}}}^{\tau, \bar{\beta}}$ for which the monodromy equations hold at all $0 \leq j' \leq f'-1$. If $\tilde{w}_j \notin \{\alpha, \text{id}\}$ for all j , then $R_{\bar{\mathfrak{M}}}^{\tau, \bar{\beta}, \nabla}$ is a power series ring over $\widehat{\otimes}_{j \in \{0, \dots, f-1\}} R_{\tilde{w}_j}^{\text{expl}, \nabla}$ where $R_{\tilde{w}_j}^{\text{expl}, \nabla}$ is as in Table 6.*

Proof. The key point is that, using ι , it suffices to impose the monodromy conditions at just the first f embeddings. Furthermore, the monodromy condition can be imposed after base change, that is, we have a cartesian diagram:

$$\begin{array}{ccc} \mathrm{Spf} R_{\overline{\mathfrak{M}}}^{\tau, \overline{\beta}, \nabla} & \longrightarrow & \mathrm{Spf} R_{\overline{\mathfrak{M}}}^{\tau, \overline{\beta}} \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Spf} R_{\overline{\mathfrak{M}}}^{\tau', \overline{\beta}, \nabla} & \longrightarrow & \mathrm{Spf} R_{\overline{\mathfrak{M}}}^{\tau', \overline{\beta}}. \end{array}$$

□

Remark 6.10. The leading term of monodromy at j is the same as in Table 5 using $\mathbf{a}'^{(j)}_k$ in place of $\mathbf{a}^{(j)}_k$. The error term will be different but this has no effect on the computations in §5.3.

7. APPLICATIONS

In this section, we apply the descriptions of the deformation rings to modularity lifting and the Serre weight conjectures. Before stating the main theorems, we describe a global setup. The particulars of the setup are not so important. The proofs of the main theorems only rely on the existence of patched modules satisfying the axioms spelled out in Definition 7.11.

7.1. Global setup. Let F/\mathbb{Q} be a CM field with maximal totally real subfield $F^+ \neq \mathbb{Q}$ and write Σ_p^+ (resp. Σ_p) for the places of F^+ (resp. of F) lying above p . Let c denote the generator of $\text{Gal}(F/F^+)$ and assume that for all places $v \in \Sigma_p^+$, v decomposes as ww^c in F .

Let $G_{/F^+}$ be a reductive group which is an outer form for GL_3 which is quasi-split at all finite places of F^+ and which splits over F . Suppose that $G(F_v^+) \cong U_3(\mathbb{R})$ for all $v|\infty$. Recall from [EGH13, §7.1] that G admits a reductive model \mathcal{G} defined over $\mathcal{O}_{F^+}[1/N]$, for some $N \in \mathbb{N}$ which is prime to p , together with an isomorphism

$$(7.1) \quad \iota : \mathcal{G}/\mathcal{O}_F[1/N] \xrightarrow{\iota} \text{GL}_3/\mathcal{O}_F[1/N]$$

which specializes to $\iota_w : \mathcal{G}(\mathcal{O}_{F_v^+}) \xrightarrow{\sim} \mathcal{G}(\mathcal{O}_{F_w}) \xrightarrow{\iota} \text{GL}_3(\mathcal{O}_{F_w})$ for all places $v \in \Sigma_p^+$.

Define $F_p^+ := F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $\mathcal{O}_{F^+,p} := \mathcal{O}_{F^+} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. If W is a finite \mathcal{O} -module endowed with a continuous action of $\mathcal{G}(\mathcal{O}_{F^+,p})$ and $U \leq G(\mathbb{A}_{F^+}^{\infty,p}) \times \mathcal{G}(\mathcal{O}_{F^+,p})$ is a compact open subgroup, the space of algebraic automorphic forms on G of level U and coefficients in W is the \mathcal{O} -module defined as:

$$(7.2) \quad S(U, W) \stackrel{\text{def}}{=} \{f : G(F^+) \backslash G(\mathbb{A}_{F^+}^{\infty}) \rightarrow W \mid f(gu) = u_p^{-1}f(g) \ \forall \ g \in G(\mathbb{A}_{F^+}^{\infty}), u \in U\}.$$

We recall that the level U is said to be *sufficiently small* if for all $t \in G(\mathbb{A}_{F^+}^{\infty})$, the finite group $t^{-1}G(F^+)t \cap U$ is of order prime to p . For a finite place v of F^+ that splits in F , we say that U is *unramified* at v if one has a decomposition $U = \mathcal{G}(\mathcal{O}_{F_v^+})U^v$ for some compact open subgroup $U^v \leq G(\mathbb{A}_{F^+}^{\infty,v})$. If w is a finite place of F we say, with an abuse, that w is an unramified place for U if its restriction $w|_{F^+}$ is unramified for U .

Let \mathcal{P}_U be the set of finite places w of F such that $v \stackrel{\text{def}}{=} w|_{F^+}$ is split in F , $v \nmid p$ and U is unramified at v . For any subset $\mathcal{P} \subseteq \mathcal{P}_U$ of finite complement that is closed under complex conjugation, we write $\mathbb{T}_{\mathcal{P}} = \mathcal{O}[T_w^{(i)}, w \in \mathcal{P}, i \in \{0, 1, 2, 3\}]$ for the universal Hecke algebra on \mathcal{P} , where the Hecke operator $T_w^{(i)}$ acts on the space $S(U, W)$ as the usual double coset operator

$$\iota_w^{-1} \left[\text{GL}_3(\mathcal{O}_{F_w}) \begin{pmatrix} \varpi_w \text{Id}_i & 0 \\ 0 & \text{Id}_{3-i} \end{pmatrix} \text{GL}_3(\mathcal{O}_{F_w}) \right].$$

The space of algebraic automorphic forms $S(U, W)$ is then endowed with an action of the Hecke algebra $\mathbb{T}_{\mathcal{P}}$.

A *Serre weight* (for \mathcal{G}) is an isomorphism class of a smooth, absolutely irreducible representation V of $\mathcal{G}(\mathcal{O}_{F^+,p})$. If $v|p$ is a place of F^+ , a *Serre weight at v* is an isomorphism class of a smooth, absolutely irreducible representation V_v of $\mathcal{G}(\mathcal{O}_{F_v^+})$. Finally, if $w|p$ is a place of F , a *Serre weight at w* is an isomorphism class of a smooth, absolutely irreducible representation V_w of $\mathrm{GL}_3(\mathcal{O}_{F_w})$. Note that if V_v is a Serre weight at a place v such that $v = ww^c$ in F , then the Serre weights at w^c defined by $V_v \circ \iota_w^{-1} \circ c$, $V_v \circ \iota_{w^c}^{-1}$ are dual to each other. Any Serre weight V for $\mathcal{G}(\mathcal{O}_{F^+,p})$ can be written as $V \cong \bigotimes_{v|p} V_v$ where V_v are Serre weights at v .

Definition 7.1. Let $\bar{r} : G_F \rightarrow \mathrm{GL}_3(\mathbb{F})$ be a continuous Galois representation and let V be a Serre weight for \mathcal{G} . We say that \bar{r} is *automorphic of weight V* (or that V is a Serre weight of \bar{r}) if there exists a compact open subset U in $G(\mathbb{A}_F^{\infty,p}) \times \mathcal{G}(\mathcal{O}_{F^+,p})$ which is unramified at places $v|p$, and a cofinite subset $\mathcal{P} \subset \mathcal{P}_U$ such that

$$S(U, V)_{\bar{\mathbf{m}}} \neq 0$$

where $\bar{\mathbf{m}}$ is the kernel of the system of Hecke eigenvalues $\bar{\alpha} : \mathbb{T}_{\mathcal{P}} \rightarrow \mathbb{F}$ associated to \bar{r} , and $\bar{\alpha}$ satisfies the equality

$$\det(1 - \bar{r}^\vee(\mathrm{Frob}_w)X) = \sum_{j=0}^3 (-1)^j (\mathbf{N}_{F_w/\mathbb{Q}_p}(w))^{(j)} \bar{\alpha}(T_w^{(j)}) X^j$$

for all $w \in \mathcal{P}$. We write $W(\bar{r})$ for the set of all Serre weights of \bar{r} . We say that \bar{r} is *automorphic* if $W(\bar{r}) \neq \emptyset$.

From now on, we assume that p is unramified and totally split in F . If $w|p$ is a place of F and $w|_{F^+} = v$ following [GHS], we write $(X_1^{(3)})_v$ to be the set of p -restricted pairs $\{\underline{a}_w, \underline{a}_{w^c}\} \subset \mathbb{Z}^3$ such that $a_{i,w} + a_{2-i,w^c} = 0$ for all $1 \leq i \leq 3$ (recall that p -restricted means that $p-1 \geq a_{i,w} - a_{i+1,w} \geq 0$ for $i \in \{1, 2\}$). To a p -restricted element $\underline{a}_w \in \mathbb{Z}^3$, we associate an irreducible representation $F_{\underline{a}_w}$ of $\mathrm{GL}_3(k_w)$ and, by inflation, $\mathrm{GL}_3(\mathcal{O}_{F_w})$ as in [GHS, §3.1] (cf. also [EGH13, (4.1.3)]). To an element $\{\underline{a}_w, \underline{a}_{w^c}\} \in (X_1^{(3)})_v$, we associate an irreducible representation $F_{\underline{a}_v} \stackrel{\mathrm{def}}{=} F_{\underline{a}_w} \circ \iota_w$ of $\mathcal{G}(\mathcal{O}_{F_v^+})$ that is independent of the choice of place w dividing v . Let $(X_1^{(3)})_0^{\Sigma_p}$ be the set of $\underline{a} = (\underline{a}_w)_{w|p}$ where $\{\underline{a}_w, \underline{a}_{w^c}\} \in (X_1^{(3)})_v$. Given an element $\underline{a} \in (X_1^{(3)})_0^{\Sigma_p}$, we associate an irreducible representation $F_{\underline{a}} \stackrel{\mathrm{def}}{=} \bigotimes_{v|p} F_{\underline{a}_v}$ of $\mathcal{G}(\mathcal{O}_{F^+,p})$, or in other words a Serre weight for \mathcal{G} . All Serre weights are of the form $F_{\underline{a}}$ for some $\underline{a} \in (X_1^{(3)})_0^{\Sigma_p}$ and $F_{\underline{a}} \cong F_{\underline{a}'}$ if and only if $\underline{a} \sim \underline{a}'$ as in Section 3.1 of [GHS].

Definition 7.2. If $w|p$ and $\underline{a}_w = (a, b, c) \in \mathbb{Z}^3$ is p -restricted, let $F(a, b, c) := F_{\underline{a}_w}$ be the corresponding weight at w . Then $F(a, b, c)$ is *lower alcove* or *Fontaine-Laffaille* if $a - c < p - 2$ and it is in the *upper alcove* if $a - c > p - 2$. We say that $F(a, b, c)$ is *generic* if

$$3 < a - b, b - c < p - 5 \text{ and } |a - c - (p - 2)| > 4.$$

An analogous notion of genericity can be introduced for $F_{\underline{a}_v}$, where $v|p$ and $\underline{a}_v \in (X_1^{(3)})_v$. Finally, if $\underline{a} \in (X_1^{(3)})_0^{\Sigma_p}$, we say that $F_{\underline{a}} = \bigotimes_{v|p} F_{\underline{a}_v}$ is generic if $F_{\underline{a}_v}$ is generic for all $v|p$. If $\bar{r} : G_F \rightarrow \mathrm{GL}_3(\mathbb{F})$ is as in Definition 7.1, we write $W_{\mathrm{gen}}(\bar{r})$ to denote the set of generic Serre weights for \bar{r} .

Recall that K is an unramified extension of \mathbb{Q}_p of degree f . We now recall the tame types for $\mathrm{GL}_3(\mathcal{O}_K)$. Let $\tau : I_K \rightarrow \mathcal{O}^\times$ be a tame inertial type. We define a $\mathrm{GL}_3(k)$ -representation $\sigma(\tau)$, valued in E , via the “inertial local Langlands correspondence” (cf. [CEG⁺], Theorem 3.7). For each tame type τ , $\sigma(\tau)$ is given in Table 2.

If $\sigma(\tau)^\circ$ is a $\mathrm{GL}_3(k)$ -stable \mathcal{O} -lattice inside $\sigma(\tau)$, we write $\mathrm{JH}(\sigma(\tau))$ to denote the set of Jordan–Hölder constituent of $\bar{\sigma}(\tau)^\circ \stackrel{\mathrm{def}}{=} \sigma(\tau)^\circ \otimes_{\mathcal{O}} \mathbb{F}$. The set $\mathrm{JH}(\sigma(\tau))$ does not depend on the choice of the lattice $\sigma(\tau)^\circ$. When $K = \mathbb{Q}_p$ and τ generic, the set $\mathrm{JH}(\sigma(\tau))$ consists of nine Serre weights which can be made quite explicit (cf. [Her], Theorem 5.1).

7.2. Modularity lifting and Serre weight conjectures. We are now ready to state our main theorems. If $r : G_F \rightarrow \mathrm{GL}_3(E)$ is a continuous Galois representation, we say (following [BLGG]) that r is *automorphic* if there exists a RACSDC representation π of $\mathrm{GL}_3(\mathbb{A}_F)$ such that $r \otimes_E \overline{\mathbb{Q}_p} \cong r_i(\pi)$ where $i : \overline{\mathbb{Q}_p} \xrightarrow{\sim} \mathbb{C}$ and $r_i(\pi) : G_F \rightarrow \mathrm{GL}_3(\overline{\mathbb{Q}_p})$ is the continuous representation attached to π by [BLGG], Theorem 2.1.2.

Definition 7.3. Let $\bar{r} : G_F \rightarrow \mathrm{GL}_3(E)$ be a continuous Galois representation. We say that \bar{r} satisfies the Taylor–Wiles conditions if

- \bar{r} has image containing $\mathrm{GL}_3(\mathbb{F}_0)$ for some $\mathbb{F}_0 \subset \mathbb{F}$ with $\#\mathbb{F}_0 > 9$.
- $\overline{F}^{\ker \mathrm{ad} \bar{r}}$ does not contain $F(\zeta_p)$.

From now on we further assume that

- the extension F/F^+ is unramified at all finite places; and
- If $\bar{r} : G_F \rightarrow \mathrm{GL}_3(\mathbb{F})$ is ramified at a place w of F , then $v = w|_{F^+}$ splits as ww^c (split ramification).

We make these two assumptions in order to construct a minimal patched module in Section 7.4. These assumptions can be removed by using a not necessarily minimal patched module, but we avoid this for ease of exposition.

Theorem 7.4. *Let $r : G_F \rightarrow \mathrm{GL}_3(E)$ be an absolutely irreducible Galois representation and write \bar{r} for the reduction of a G_F -stable \mathcal{O} -lattice in r .*

Assume that:

- (1) p splits completely in F^+ ;

- (2) r is conjugate self-dual and unramified almost everywhere;
- (3) for all places $w \in \Sigma_p$, the representation $r|_{G_{F_w}}$ is potentially crystalline, with parallel Hodge type $(2, 1, 0)$ and with strongly generic tame inertial type $\tau_{\Sigma_p^+} = \otimes_{v \in \Sigma_p^+} \tau_v$ (cf. Definition 2.1);
- (4) \bar{r} verifies the Taylor-Wiles conditions (cf. Definition 7.3) and \bar{r} has split ramification;
- (5) $\bar{r} \cong \bar{r}_i(\pi)$ for a RACSDC representation π of $\mathrm{GL}_3(\mathbb{A}_F)$ with trivial infinitesimal character such that $\otimes_{v \in \Sigma_p^+} \sigma(\tau_v)$ is a K -type for $\otimes_{v \in \Sigma_p^+} \pi_v$.

Then r is automorphic.

Remark 7.5. Note that we do not make any potentially diagonalizability assumption. In fact, we do not know whether or not $r|_{G_{F_w}}$ as in the theorem is potentially diagonalizable. We also do not assume that the residual representations have any particular form at p .

Remark 7.6. The first assumption and the strong genericity condition can both be relaxed if one assumes that at each place w the shape is not one of $\{\alpha, \beta, \gamma, \mathrm{id}\}$. The difficulty comes from the absence of a general, explicit description of the deformation ring (rather than its special fiber) in those cases, where we need the Serre weight conjectures as input to show Theorem 7.7.

Theorem 7.4 is a consequence of the following Theorem using standard Kisin-Taylor-Wiles patching methods:

Theorem 7.7. *Let $\bar{\rho} : G_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_3(\mathbb{F})$ be a continuous automorphic Galois representation. Let τ be a generic tame inertial type. If $\mathbf{w}(\bar{\rho}, \tau) \in \{\alpha, \beta, \gamma, \mathrm{id}\}$, assume furthermore that τ is strongly generic. Then the framed potentially crystalline deformation ring $R_{\bar{\rho}}^{(2,1,0),\tau}$ with Hodge-Tate weights $(2, 1, 0)$ has connected generic fiber.*

If $\bar{\rho} : G_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_3(\mathbb{F})$ is a continuous semisimple Galois representation an explicit set of weights $W^?(\bar{\rho}|_{I_{\mathbb{Q}_p}})$ is defined in [Her09, Conjecture 6.9]. The main conjecture in *loc. cit.* is that $W^?(\bar{\rho}|_{I_{\mathbb{Q}_p}})$ should give the set of modular weights. More precisely, fix a place \tilde{v} above each $v \in \Sigma_p^+$, we prove the following generalization of the weight part of Serre's conjecture as conjectured in [Her09, Conjecture 6.9] (cf. §7.4):

Theorem 7.8. *Let $\bar{r} : G_F \rightarrow \mathrm{GL}_3(\mathbb{F})$ be a continuous Galois representation, verifying the Taylor-Wiles conditions. Assume that $\bar{r}|_{G_{F_{\tilde{v}}}}$ is semisimple and 6-generic for all $v \in \Sigma_p^+$, that \bar{r} is automorphic of some generic Serre weight, and that \bar{r} has split ramification outside p . Then*

$$\bigotimes_{v \in \Sigma_p^+} F_{\underline{a}_v} \in W_{\mathrm{gen}}(\bar{r}) \iff F_{\underline{a}_v} \circ \iota_{\tilde{v}}^{-1} \in W^?(\bar{r}|_{I_{F_{\tilde{v}}}}) \quad \text{for all } v \in \Sigma_p^+.$$

Remark 7.9. Theorem 7.8 is stated only for $\bar{\tau}$ which are semisimple above p because those are the only representations for which there is an explicit conjecture. Our computations together with work of [HLM], [MP] suggest a set $W^?(\bar{\rho})$ for non-semisimple $\bar{\rho}$ for which the analogue of Theorem 7.8 should hold. One example is worked out in Proposition 7.17.

Remark 7.10. The restriction to generic weights in the statement of Theorem 7.8 was due to the current weight elimination results. Recent improvements ([HLM], [MP], [LMP]) show that $W_{gen}(\bar{\tau})$ can be replaced by $W(\bar{\tau})$ and ‘automorphic of some generic Serre weight’ with just ‘automorphic’ in most cases. Specifically, [HLM] §2.5 and [MP] deals with the niveau 1 case (cf. [LMP] §3 when $\bar{\rho}$ has niveau 2). Recent unpublished work from current Ph.D. student John Enns conveyed to us by private correspondence handles the case when $\bar{\tau}$ is irreducible at places above p and 8-generic.

The proof uses the Breuil-Mézard philosophy introduced in [GK14]. Namely, we use the descriptions of the special fibers of deformation rings to determine the Hilbert-Samuel multiplicities of minimal patched modules. Assuming first that $\bar{\tau}$ is modular of a Fontaine-Laffaille weight, we use an inductive argument involving carefully chosen tame types to prove modularity of the shadow weights (see Proposition 7.16). A slightly more intricate argument shows that if $\bar{\tau}$ is modular, then it is modular of a Fontaine-Laffaille weight.

7.3. Weak minimal patched modules. Let $\bar{\tau} : G_F \rightarrow \mathrm{GL}_3(\mathbb{F})$ be a Galois representation. For each place $v|p$ of F^+ , fix a place \tilde{v} of F such that $\tilde{v}|_{F^+} = v$. Let R_v^\square denote the unrestricted universal \mathcal{O} -framed deformation ring of $\bar{\tau}|_{G_{F_{\tilde{v}}}}$. Fix a natural number h and let

$$R_\infty = \left(\widehat{\bigotimes_{v \in \Sigma_p^+} R_v^\square} \right) \llbracket x_1, x_2, \dots, x_h \rrbracket \text{ and } X_\infty = \mathrm{Spf} R_\infty.$$

If $\tau_{\tilde{v}}$ is an inertial type for $G_{F_{\tilde{v}}}$, then let $R_v^{\square, \tau_{\tilde{v}}}$ be the universal \mathcal{O} -framed deformation ring of $\bar{\tau}|_{G_{F_{\tilde{v}}}}$ of type $\tau_{\tilde{v}}$. If $\tau = \bigotimes_{v \in \Sigma_p^+} \tau_{\tilde{v}}$, then let

$$R_\infty(\tau) = \left(\widehat{\bigotimes_{v \in \Sigma_p} R_v^{\square, \tau_{\tilde{v}}}} \right) \llbracket x_1, x_2, \dots, x_h \rrbracket \text{ and } X_\infty(\tau) = \mathrm{Spf} R_\infty(\tau).$$

Let $d+1$ be the dimension of $R_\infty(\tau)$ (the dimension is independent of τ by Theorem 3.3.4 of [Kis08]). We denote by \overline{R}_v^\square , \overline{R}_∞ , etc. the reduction of these objects modulo ϖ . The following definition is adapted from Definition 4.1.1 of [GHS].

Definition 7.11. A *weak minimal patching functor* for $\bar{\tau}$ is defined to be a covariant exact functor $M_\infty : \mathrm{Rep}_K(\mathcal{O}) \rightarrow \mathrm{Mod}(X_\infty)$ satisfying the following axioms:

- (1) Let $\tau \stackrel{\text{def}}{=} \bigotimes_{v \in \Sigma_p^+} \tau_v$, where for all $v \in \Sigma_p^+$, τ_v is an inertial type, and let $\sigma(\tau) \stackrel{\text{def}}{=} \bigotimes_{v \in \Sigma_p^+} \sigma(\tau_v)$ be the associated K -type as in [CEG⁺, Theorem 3.7]. If $\sigma(\tau)^\circ$ an \mathcal{O} -lattice in it, then $M_\infty(\sigma(\tau)^\circ)$ is p -torsion free and is maximally Cohen-Macaulay over $R_\infty(\tau)$;
- (2) if $V = \bigotimes_{v \in \Sigma_p^+} V_v$, where for all $v \in \Sigma_p^+$ the V_v are irreducible $\mathcal{G}(k_v)$ -representations over \mathbb{F} , the module $M_\infty(V)$ has nonempty support of dimension d if $\bar{\tau}$ is automorphic of weight V and is 0 otherwise; and
- (3) the locally free sheaf $M_\infty(\sigma(\tau)^\circ)[1/p]$ (being maximal Cohen-Macaulay over the regular generic fiber of $X_\infty(\tau)$) has rank at most one on each connected component.

Remark 7.12. The adjective “weak” corresponds to the fact that $M_\infty(\sigma(\tau)^\circ)$ is not assumed to have full support on $X_\infty(\tau)$ for all inertial types τ in contrast to Definition 4.1.1 of [GHS].

Remark 7.13. The adjective “minimal” corresponds to the multiplicity one property in condition (3). Our results on automorphy of global Serre weights could be proved without requiring minimality using the geometric perspective of [EG14], but we have avoided this for ease of exposition.

Given a Noetherian ring R and an R -module M , we denote the Hilbert-Samuel multiplicity of M by $e(M, R)$. If $R = R_\infty$, let $e(M) = e(M, R_\infty)$. The following proposition is the key to relating automorphy of global Serre weights to multiplicities of deformation rings.

Proposition 7.14. *If M_∞ is a weak minimal patching functor, then $e(M_\infty(\sigma(\bar{\tau}))) \leq e(\bar{R}_\infty(\tau))$ and we have equality if and only if $M_\infty(\sigma(\tau)^\circ)$ has full support on $X_\infty(\tau)$ (for any choice of lattice $\sigma(\tau)^\circ$).*

Proof. Let $\mathbb{T}_\infty(\tau)$ be the quotient of $R_\infty(\tau)$ which acts faithfully on $M_\infty(\sigma(\tau)^\circ)$. Then

$$e(M_\infty(\sigma(\bar{\tau})^\circ)) = e(\bar{\mathbb{T}}_\infty(\tau)) \leq e(\bar{R}_\infty(\tau))$$

where the equality follows from Definition 7.11(3) and Corollary 1.3.5 of [Kis09a] and the inequality follows from the fact that $\dim \mathbb{T}_\infty(\tau) = \dim R_\infty(\tau)$ by Definition 7.11(1). The inequality is an equality if and only if $\mathbb{T}_\infty(\tau) = R_\infty(\tau)$ since $R_\infty(\tau)$ is reduced and equidimensional. \square

We now construct a weak minimal patching functor for $\bar{\tau}$ under some hypotheses using the Taylor-Wiles method. We write Σ_0 to denote the finite primes of F where $\bar{\tau}$ ramifies and define $\Sigma_0^+ \stackrel{\text{def}}{=} \{w|_{F^+}, w \in \Sigma_0\}$ and $\Sigma^+ \stackrel{\text{def}}{=} \Sigma_p^+ \cup \Sigma_0^+$. Assume for the rest of this section that $\bar{\tau}$ satisfies the Taylor-Wiles conditions of Definition 7.3. Note that the first condition, which is stronger than the usual condition of adequacy, allows us (see Section 2.3 of [CEG⁺]) to choose a place $v_1 \notin \Sigma^+$ of F^+ such that

- v_1 splits in F as $v_1 = \tilde{v}_1 \tilde{v}_1^c$;
- v_1 does not split completely in $F(\zeta_p)$; and
- $\bar{\rho}(\text{Frob}_{w_1})$ has distinct \mathbb{F} -rational eigenvalues, no two of which have ratio $(\mathbf{N}_{F^+/\mathbb{Q}} v_1)^{\pm 1}$.

In order to satisfy the minimality condition, recall (cf. §7.2) that we have made the following two further assumptions.

- The extension F/F^+ is unramified at all finite places.
- If $\bar{\tau} : G_F \rightarrow \text{GL}_3(\mathbb{F})$ is ramified at a place w of F , then $v = w|_{F^+}$ splits as ww^c .

As mentioned in Remark 7.13, these two assumptions can be removed by working with weak patched modules which are not necessarily minimal. For $v \in \Sigma_0^+$, let τ_v be the type which is minimally ramified with respect to $\bar{\tau}|_{G_{F_w}}$ (τ_v is the restriction to inertia of the Weil-Deligne representation attached to a Galois representation which is minimal in the sense of Definition 2.4.14 of [CHT08]). Let R_v^{\square, τ_v} be the corresponding universal \mathcal{O} -framed deformation ring of $\bar{\tau}|_{G_{F_v}}$. Let $R_{v_1}^{\square}$ be the unrestricted universal \mathcal{O} -framed deformation ring of $\bar{\tau}|_{G_{F_{v_1}}}$. Let

$$R^{\text{loc}} = \widehat{\bigotimes_{v \in \Sigma_p^+} R_v^{\square}} \widehat{\bigotimes_{v \in \Sigma_0^+} R_v^{\square, \tau_v}} \widehat{\bigotimes_{v \in \Sigma_0^+} R_{v_1}^{\square}}.$$

Choose an integer $q \geq 3[F^+ : \mathbb{Q}]$ as in Section 2.5 of [CEG⁺], and let

$$R_{\infty} = R^{\text{loc}}[[x_1, \dots, x_{q-3[F^+ : \mathbb{Q}]}]].$$

By [CHT08, Corollary 2.4.21], $R_v^{\square, \tau}$ is formally smooth over \mathcal{O} for $v \in \Sigma_0^+$. By the choice of v_1 , $R_{v_1}^{\square}$ is formally smooth over \mathcal{O} by [CEG⁺, Proposition 2.5]. We conclude that R_{∞} is formally smooth over $\widehat{\bigotimes_{v \in \Sigma_p^+} R_v^{\square}}$ and hence there is an isomorphism $R_{\infty} \cong \widehat{\bigotimes_{v \in \Sigma_p^+} R_v^{\square}}[[x_1, x_2, \dots, x_h]]$ for some natural number h .

One can construct a $R_{\infty}[[G(\mathcal{O}_{F^+, p})]]$ -module M_{∞} as in [Le, Section 4.2] (p is assumed to split in F in [Le], however the construction, results, and proofs of Section 4 extend verbatim). Then define a covariant functor $M_{\infty} : \text{Rep}_{\mathcal{G}(\mathcal{O}_{F^+, p})}(\mathcal{O}) \rightarrow \text{Mod}(X_{\infty})$ by $M_{\infty}(W) = \text{Hom}_{\mathcal{G}(\mathcal{O}_{F^+, p})}(W, M_{\infty}^{\vee})^{\vee}$ where \cdot^{\vee} denotes the Pontriagin dual.

Proposition 7.15. *If E is sufficiently large, M_{∞} is a weak minimal patching functor.*

Proof. This proof is adapted from various proofs in [CEG⁺] and [Le]. While the contexts differ slightly, the proofs apply verbatim. Note that the definition of $M_{\infty}(\sigma(\tau)^{\circ})$ agrees with the definition given after 4.13 of [CEG⁺] by [CEG⁺, Remark 4.15]. This definition guarantees that $M_{\infty}(\sigma(\tau)^{\circ})$ is p -torsion free. Exactness of M_{∞} follows from [CEG⁺, Proposition 2.10] (the choice of place v_1 guarantees projectivity). 1 and 3 are proved similarly to [CEG⁺, Lemma 4.18(1)].

If V is a global Serre weight, by Theorem 5.2.1(iii) of [HLM] and Nakayama's lemma (using Theorem 5.2.1(i) of [HLM]), $M_\infty(V)$ is nonzero if and only if $\bar{\tau}$ is automorphic of weight V . By Theorem 4.1.4(2) of [Le], $M_\infty(V)$ is maximal Cohen-Macaulay of depth d , which shows 2. \square

7.4. Shapes and Serre weights. We now prove Theorem 7.8. The key ingredient in the proof of Theorem 7.8 is the description of the deformation rings in Table 6 matched with the conjectural weights $W^?(\bar{\rho})$. We first recall the notion of shadow weight. For $\bar{\rho} : G_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_3(\bar{\mathbb{F}})$ semisimple and generic, $W^?(\bar{\rho})$ contains six obvious weights $W_{\mathrm{obv}}(\bar{\rho})$ (three upper alcove weights and three lower alcove weights, cf. Definition 7.1.4 [GHS]). Each $F(a, b, c) \in W_{\mathrm{obv}}(\bar{\rho})$ which is lower alcove has a corresponding *shadow weight* $F(p - 2 + c, b, a - p + 2) \in W^?(\bar{\rho})$ (cf. Definition 7.2.3 [GHS]). A combinatorial description of $W_{\mathrm{obv}}(\bar{\rho})$ can be found e.g. in [BLGG], Lemma 5.1.2 (where $W_{\mathrm{obv}}(\bar{\rho})$ is denoted by $W^{\mathrm{explicit}}(\bar{\rho})$) or in [Her09], Proposition 6.28 and Lemma 7.6. For the convenience of the reader, we include the set of predicted weights $W^?(\bar{\rho})$ in Table 9.

The following combinatorial result matches shapes with predicted Serre weights.

Proposition 7.16. *Let $\bar{\rho} : G_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_3(\mathbb{F})$ be semisimple and 6-generic.*

- (1) *If $F(a, b, c) \in W_{\mathrm{obv}}(\bar{\rho}|_{I_{\mathbb{Q}_p}})$, then there is a generic tame type τ such that $\mathrm{JH}(\sigma(\tau)) \cap W^?(\bar{\rho}|_{I_{\mathbb{Q}_p}}) = \{F(a, b, c)\}$ and $\mathbf{w}(\bar{\rho}, \tau)$ has length 4.*
- (2) *If $F(p - 2 + c, b, a - p + 2) \in W^?(\bar{\rho}|_{I_{\mathbb{Q}_p}})$ is a shadow weight, then there is a generic tame type τ such that $\mathrm{JH}(\sigma(\tau)) \cap W^?(\bar{\rho}|_{I_{\mathbb{Q}_p}}) = \{F(a, b, c), F(p - 2 + c, b, a - p + 2)\}$ and $\mathbf{w}(\bar{\rho}, \tau)$ is a length 3 shadow shape.*
- (3) *If $F(a, b, c) \in W^?(\bar{\rho}|_{I_{\mathbb{Q}_p}})$ is an obvious upper (resp. lower) alcove weight, then there is generic tame type τ such that $\mathrm{JH}(\sigma(\tau)) \cap W^?(\bar{\rho}|_{I_{\mathbb{Q}_p}})$ contains $F(a, b, c)$ plus one other obvious lower (resp. upper) alcove weight and $\mathbf{w}(\bar{\rho}, \tau)$ is a length 3 non-shadow shape.*
- (4) *If $F(p - 2 + c, b, a - p + 2) \in W^?(\bar{\rho}|_{I_{\mathbb{Q}_p}})$ is a shadow weight, then there is a generic tame type τ such that $\mathrm{JH}(\sigma(\tau)) \cap W^?(\bar{\rho}|_{I_{\mathbb{Q}_p}})$ contains $F(p - 2 + c, b, a - p + 2)$ plus three obvious weights of $W^?(\bar{\rho}|_{I_{\mathbb{Q}_p}})$ and $\mathbf{w}(\bar{\rho}, \tau)$ has length 2.*

Proof. The strategy is the same in all four cases so we focus on proof of (2). The type τ for (2) is given in Table 7. The table is constructed starting with a tame type τ over \mathbb{Q}_p . For each shape $\tilde{w} \in \{\alpha\beta\alpha, \gamma\beta\gamma, \alpha\gamma\alpha\}$, one can consider the mod p Kisin modules of shape \tilde{w} as in Table 3. We consider the special Kisin modules of this shape where the Frobenius $A_{\tilde{w}}$ is a monomial matrix.

For example, when $\tau = \omega^{-a} \oplus \omega^{-b} \oplus \omega^{-c}$, the Kisin module with Frobenius

$$A_{\alpha\beta\alpha} = \begin{pmatrix} 0 & 0 & v\bar{c}_{13}^* \\ 0 & v\bar{c}_{22}^* & 0 \\ v\bar{c}_{31}^* & 0 & 0 \end{pmatrix}$$

corresponds (under T_{dd}^*) to $\bar{\rho}|_{I_{\mathbb{Q}_p}} = \omega^{b+1} \oplus \text{Ind}(\omega_2^{(a+1)+p(c+1)})$. A similar computation can be done for the other shapes and cuspidal types using Proposition 2.27. We give another example below. Each semisimple $\bar{\rho}$ arises in this way from exactly three types. Comparing $\text{JH}(\sigma(\tau))$ and $W^?(\bar{\rho}|_{I_{\mathbb{Q}_p}})$ is a tedious but not difficult computation. We see that in each case the intersection $\text{JH}(\sigma(\tau)) \cap W^?(\bar{\rho}|_{I_{\mathbb{Q}_p}})$ is exactly a lower alcove weight together with its shadow.

Briefly, regarding parts (1), (3), and (4), for each tame type τ , there are six shapes of length 4, six non-shadow shapes of length 3, and six shapes of length 2. For each semisimple $\bar{\rho}$, there are exactly six types for which $\mathbf{w}(\bar{\rho}, \tau)$ has length 4, six types for which $\mathbf{w}(\bar{\rho}, \tau)$ has non-shadow length 3 and six types for which $\mathbf{w}(\bar{\rho}, \tau)$ has length 2. For each obvious weight, there is a unique type satisfying (1) and two types satisfying (3). For each shadow weight, there are two types which work for part (4).

As another example, consider $\sigma(\tau) = \text{Ind}_{P_1(\mathbb{F}_p)}^{\text{GL}_3(\mathbb{F}_p)}(\tilde{\omega}^{b+1} \otimes \Theta(\tilde{\omega}_2^{(c-1)+pa}))$ and parallel shape $\beta\gamma\beta$ which appears in the third section of Table 7. Define $\tau' \stackrel{\text{def}}{=} \omega_2^{(b+1)+p(b+1)} \oplus \omega_2^{(c-1)+pa} \oplus \omega_2^{a+p(c-1)}$ which has the orientation $s'_0 = (12)$ and $s'_1 = (132)$. The special point of $(\overline{\mathfrak{M}}, \iota) \in Y^{\mu, \tau}(\mathbb{F})$ of shape $\beta\gamma\beta$ has Frobenius given by

$$A_{\beta\gamma\beta}^{(j)} = \begin{pmatrix} 0 & \bar{c}_{12}^* & 0 \\ \bar{c}_{21}^* v^2 & 0 & 0 \\ 0 & 0 & \bar{c}_{33}^* v \end{pmatrix}$$

We are only interested in the restriction to inertia and so we can forget ι and set the constants to 1. Let $\overline{\mathcal{M}} \stackrel{\text{def}}{=} \overline{\mathfrak{M}} \otimes_{\mathbb{F}[[u]]} \mathbb{F}((u)) \in \Phi\text{-Mod}_{dd}^{\text{ét}}(\mathbb{F})$ be the associated étale φ -module with descent datum. By Proposition 2.27, the étale $(\varphi^2, k' \otimes_{\mathbb{F}_p} \mathbb{F}((v)))$ -module $\overline{\mathcal{M}}_0 \stackrel{\text{def}}{=} \varepsilon_0((\overline{\mathcal{M}})^{\Delta'=\text{id}})$ is described, in an appropriate basis $\mathfrak{f} \stackrel{\text{def}}{=} (e_1, e_2, e_3)$, as follows:

$$\begin{aligned} \text{Mat}(\phi_{\mathcal{M}}) &= \left(s'_0 A^{(1)} \begin{pmatrix} v^{c-1} & 0 & 0 \\ 0 & v^{b+1} & 0 \\ 0 & 0 & v^a \end{pmatrix} s'_0 \right) \cdot \varphi \left(s'_1 A^{(0)} \begin{pmatrix} v^{c-1} & 0 & 0 \\ 0 & v^{b+1} & 0 \\ 0 & 0 & v^a \end{pmatrix} (s'_1)^{-1} \right) \\ &= \begin{pmatrix} 0 & 0 & v^{(b+1)+p(c+1)} \\ v^{(c+1)+p(a+1)} & 0 & 0 \\ 0 & v^{(a+1)+p(b+1)} & 0 \end{pmatrix} \end{aligned}$$

up to constants. In particular, we see that the $\varphi^{3f'}$ -action on e_1 is described by

$$e_1 \mapsto v^{(p^3+1)((b+1)+p(c+1)+p^2(a+1))} e_1.$$

We conclude that $\bar{\rho} := T_{d\ell}^*(\overline{\mathfrak{M}})$ is tame with

$$\bar{\rho}|_{I_{\mathbb{Q}_p}} \cong \omega_3^{(a+1)+p(b+1)+p^2(c+1)} \oplus \omega_3^{(b+1)+p(c+1)+p^2(a+1)} \oplus \omega_3^{(c+1)+p(a+1)+p^2(b+1)}.$$

We conclude that (up to unramified twist) $\bar{\rho} = \text{Ind}_{G_{\mathbb{Q}_{p^3}}}^{G_{\mathbb{Q}_p}} \omega_3^{(a+1)+p(b+1)+p^2(c+1)}$ and so $\mathbf{w}(\bar{\rho}, \tau) = \beta\gamma\beta$. The computation of the étale $(\varphi, k' \otimes_{\mathbb{F}_p} \mathbb{F}((v)))$ -modules in the other cases is similar and left as an exercise to the reader. \square

Proof of Theorem 7.8. For generic tame types τ_v and $\bar{\rho}_v := \bar{\tau}|_{G_{F_{\bar{v}}}}$, if $\mathbf{w}(\bar{\rho}_v, \tau_v)$ has length greater than or equal 2 for all $v \mid p$, then by Theorem 6.9 and Table 6, $R_{\infty}(\tau)$ has connected generic fiber and so if $M_{\infty}(\sigma(\tau)^{\circ})$ is non-zero, then it has full support. By Proposition 7.14, we know the Hilbert-Samuel multiplicity of $M_{\infty}(\bar{\sigma}(\tau)^{\circ})$. The strategy is to compute $e(M_{\infty}(\otimes_{v \in \Sigma_p^+} V_v))$ by varying the tame type. In fact, we show that $e(M_{\infty}(\otimes_{v \in \Sigma_p^+} V_v)) = 1$ whenever $V_v \in W^?(\bar{\rho}_v)$ for all v .

First, assume that $\bar{\tau}$ is modular of weight $\otimes_{v \in \Sigma_p^+} V_v$ such that all V_v are Fontaine-Laffaille. Under these hypotheses, the modularity of the obvious weights is known by [BLGG, Theorem 5.1.4] so we focus on the shadow weights. Choose $V_v \in W^?(\bar{\rho}_v)$ for each $v \in \Sigma_p^+$. Let $S_p^+ \subset \Sigma_p^+$ be the set of places for which V_v is a shadow weight. For each $v \in S_p^+$, let V'_v denote the Fontaine-Laffaille weight corresponding to V_v . We induct on the size of S_p^+ . If S_p^+ is empty, then by Proposition 7.16(1), we can choose types τ_v for each $v \in \Sigma_p^+$ such that $\text{JH}(\sigma(\tau_v)) \cap W^?(\bar{\rho}_v) = \{V_v\}$. In this case, the deformation ring is a power-series ring and so $e(M_{\infty}(\otimes_{v \in \Sigma_p^+} V_v)) = 1$.

In general, for each $v \in \Sigma_p^+ \setminus S_p^+$, we choose τ_v as in 7.16(1). For each $v \in S_p^+$, we choose τ_v as in 7.16(2) to contain exactly V_v and V'_v . Consider $M_{\infty}(\bar{\sigma}(\tau))$ over $\bar{R}_{\infty}(\tau)$. The deformation rings for $v \notin S_p^+$ are again formally smooth. By assumption, $\bar{\tau}$ is modular of the obvious weight and so $M_{\infty}(\bar{\sigma}(\tau))$ is nonzero. By Table 6, we deduce that

$$e(M_{\infty}(\bar{\sigma}(\tau))) = e(\bar{R}_{\infty}(\tau)) = 2^{|S_p^+|}.$$

By the inductive hypothesis, the multiplicity of any Serre weight of $\bar{\tau}$ which is a shadow in fewer embeddings is 1. We deduce that $e(M_{\infty}(\otimes_v V_v)) = 1$ and hence $\otimes_v V_v$ is modular.

Finally, we show that if $\bar{\tau}$ is modular of any generic weight then it is modular of a Fontaine-Laffaille weight. First, using Proposition 7.16(1), we note that if $\otimes_v V_v$ is a Serre weight of $\bar{\tau}$ such that V_v are all obvious weights then $e(M_{\infty}(\otimes_v V_v)) \leq 1$. Assume $\bar{\tau}$ is modular of generic weight $\otimes_v V_v$. As above, let $S_p^+ \subset \Sigma_p^+$ be the set of places for which V_v is a shadow weight. Assume S_p^+ is non-empty as this is the trickier case (the other case can be handled using Proposition 7.16(3)). Choosing types τ_v as above, we conclude that $e(M_{\infty}(\bar{\sigma}(\tau))) = 2^{|S_p^+|}$ and so

$$e(M_{\infty}(\otimes_v V_v)) \leq 2^{|S_p^+|}.$$

If the inequality is strict, then there must be a modular weight which is a shadow at fewer places and so we can repeat the argument with that weight. If not, choose a place $v_0 \in S_p^+$, and replace τ_{v_0} by a tame type τ'_{v_0} for the shadow weight V_{v_0} as in Proposition 7.16(4). For the new type τ' , we have

$$e(M_\infty(\overline{\sigma}(\tau'))) = 2^{|S_p^+|+1}.$$

By comparing multiplicities, we conclude that $M_\infty(\otimes_v V'_v) \neq 0$ for some Serre weight $\otimes_v V'_v$ of $\overline{\tau}$ such that V'_{v_0} is an obvious weight and such that $\otimes_v V'_v$ is a shadow weight in fewer embeddings. \square

We end this section with an example of non-semisimple $\overline{\rho}$.

Proposition 7.17. *Let τ be a strongly generic tame type. Let $\overline{\rho}$ be a non-semisimple representation such that $\mathbf{w}(\overline{\rho}, \tau) = \alpha$ (i.e., $\overline{c}'_{22} \neq 0$ in Table 3). Then there is a subset $W^?(\overline{\rho}) \subset W^?(\overline{\rho}^{ss})$ consisting of six weights for which the weight part of Serre's conjecture holds.*

Proof. We focus on the case of principal series type as the other cases are similar. Let $\tau = \omega^{-a} \oplus \omega^{-b} \oplus \omega^{-c}$ with $a - b, b - c > 3$. Let $\overline{\rho}$ be the unique non-split representation of $G_{\mathbb{Q}_p}$ of the form

$$\overline{\rho} \cong \begin{pmatrix} \chi_1 \omega^{a+1} & * & 0 \\ 0 & \chi_2 \omega^{b+1} & 0 \\ 0 & 0 & \chi_3 \omega^{c+1} \end{pmatrix}$$

for fixed unramified characters χ_i . Then $\mathbf{w}(\overline{\rho}, \tau) = \alpha$ (cf. Table 8).

For a fixed place $v \mid p$, let $\overline{\tau} : G_F \rightarrow \mathrm{GL}_3(\overline{\mathbb{F}})$ be such that $\overline{\tau}|_{G_{F_v}} \cong \overline{\rho}$ and such that $\overline{\tau}$ is modular of a generic Fontaine-Laffaille weight $\otimes_{v' \in \Sigma_p^+} V_{v'}$. There exists such a globalization (after possibly enlarging the field F) by Proposition 8.5 [EG14]. Note that $\overline{\rho}$ admits a Fontaine-Laffaille and hence a potentially diagonalizable lift with Hodge-Tate weights $(a+1, b+1, c+1)$. For all $v' \neq v$, fix a type $\tau_{v'}$ such that $\sigma(\tau_{v'})$ contains exactly $V_{v'}$ among $W^?((\overline{\tau}|_{G_{F_{v'}}})^{ss})$.

Let

$$W^?(\overline{\rho}) = \left\{ \begin{array}{l} F(a-1, b, c+1), F(p-2+c, a, b+1), F(a-1, c, b-p+2), \\ F(p-1+b, a, c), F(c+p-1, b, a-p+1), F(a, c, b-p+1) \end{array} \right\} \subset W^?(\overline{\rho}^{ss}).$$

The top row are obvious weights for $\overline{\rho}^{ss}$; the bottom row are shadow weights. We want to show that, for $V_{v'}$ as above, $\overline{\tau}$ is modular of all weights of the form $F_v \otimes (\otimes_{v' \neq v} V_{v'})$ with $F_v \in W^?(\overline{\rho})$. Weight elimination, i.e., that $F_v \in W^?(\overline{\rho})$, is a consequence of [MP].

The proof of modularity of five of the weights is exactly as in Theorem 7.8. The only difference is for the weight $F(a, c, b-p+1)$ which is the shadow of $F(b-1, c, a-p+2)$ which is *not* a modular weight for $\overline{\rho}$ although it is for $\overline{\rho}^{ss}$. For this weight, we consider the cuspidal type $\tau_v =$

$\omega_3^{-(a+1)-p(b-1)-p^2c} \oplus \omega_3^{-(b-1)-pc-p^2(a+1)} \oplus \omega_3^{-c-p(a+1)-p^2(b-1)}$. This is chosen such that $\text{JH}(\sigma(\tau_v)) \cap W^?(\bar{\rho}) = \{F(p-2, a, b+1), F(a, c, b-p+1)\}$ (the intersection with $W^?(\bar{\rho}^{ss})$ contains 4 weights as in Proposition 7.16(4)).

The claim is that $\mathbf{w}(\bar{\rho}, \tau)$ is a length 3 non-shadow shape. Hence, the deformation ring at v has Hilbert-Samuel multiplicity 2 which implies that $F(a, c, b-p+1)$ is modular since $F(p-2, a, b+1)$ is Fontaine-Laffaille. For the cuspidal type τ_v , consider all $\bar{\mathfrak{M}} \in Y^{\mu, \tau}(\mathbb{F})$ of (parallel) shape $\alpha\beta\gamma$. For $0 \leq j \leq 2$, we have

$$A_{\alpha\beta\gamma}^{(j)} = \begin{pmatrix} v^2 \bar{c}_{11}^* & 0 & 0 \\ v \bar{c}_{21} + v^2 \bar{c}'_{21} & 0 & \bar{c}_{23}^* \\ v^2 \bar{c}_{31} & v \bar{c}_{32}^* & 0 \end{pmatrix}$$

by Theorem 2.22. Consider a Kisin module $\bar{\mathfrak{M}}$ with $\bar{c}_{31} = \bar{c}_{21} = 0$ but $\bar{c}'_{21} \neq 0$. We set the starred entries to be 1 for simplicity, as they don't effect the restriction to inertia. By Lemma 2.21 which allows row operations, there is another eigenbasis for $\bar{\mathfrak{M}}$ such that

$$A_{\alpha\beta\gamma}^{(j)} = \begin{pmatrix} 0 & 0 & 1 \\ v^2 & 0 & (c'_{21})^{-1} \\ 0 & v & 0 \end{pmatrix}.$$

By Proposition 6.1, the orientation for the type τ'_v is $s'_0 = (123)$, $s'_1 = (123)^2$ and $s'_2 = \text{id}$. Using Proposition 2.27, we compute the matrix for the étale ϕ -module $\varepsilon_0(\mathcal{M}^{\Delta=1})$ as

$$\begin{aligned} \text{Mat}(\phi_{\mathcal{M}}) &= \begin{pmatrix} 0 & 0 & v^{c+1} \\ v^{a+1} & 0 & 0 \\ c_{21}^{-1} v^{a+1} & v^{b+1} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & v^{(a+1)p} \\ v^{(b+1)p} & 0 & c_{21}^{-1} v^{(a+1)p} \\ 0 & v^{(c+1)p} & 0 \end{pmatrix} \begin{pmatrix} 0 & c_{21}^{-1} v^{(a+1)p^2} & v^{(b+1)p^2} \\ v^{(c+1)p^2} & 0 & 0 \\ 0 & v^{(a+1)p^2} & 0 \end{pmatrix} \\ &= \begin{pmatrix} v^{(c+1)(1+p+p^2)} & 0 & 0 \\ 0 & v^{(a+1)(1+p+p^2)} & 0 \\ 0 & c_{21}^{-1} \star & v^{(b+1)(1+p+p^2)} \end{pmatrix} \end{aligned}$$

for some $\star \in \mathbb{F}((v))$. Thus, for any such Kisin module $\bar{\mathfrak{M}}$, we have

$$T_{dd'}^*(\bar{\mathfrak{M}}) |_{I_{\mathbb{Q}_p}} \cong \begin{pmatrix} \omega^{a+1} & \star & 0 \\ 0 & \omega^{b+1} & 0 \\ 0 & 0 & \omega^{c+1} \end{pmatrix}.$$

There is a unique such extension up to isomorphism and so we conclude that $\mathbf{w}(\bar{\rho}, \tau) = \alpha\beta\gamma$. \square

8. THE α AND id SHAPES

The aim of this section is complete the proof of Theorems 7.4 and 7.7 by studying the deformation rings in the most complicated cases when the shape has length 0 or 1. To recall the assumptions, τ will be a generic tame inertial type and $\bar{\rho} : G_{\mathbb{Q}_p} \rightarrow \text{GL}_3(\mathbb{F})$ satisfies $\mathbf{w}(\bar{\rho}, \tau) \in \{\alpha, \text{id}\}$. There are three cases to consider: the identity shape $\mathbf{w}(\bar{\rho}, \tau) = \text{id}$ (§ 8.1), the case where $\mathbf{w}(\bar{\rho}, \tau) = \alpha$ and $\bar{\rho}$ is semisimple (§ 8.2.1) and the case where $\mathbf{w}(\bar{\rho}, \tau) = \alpha$ and $\bar{\rho}$ is non-semisimple (§ 8.2.2). In the three cases, the cardinality of the intersection $\text{JH}(\sigma(\tau)) \cap W^?(\bar{\rho})$ is 6, 6 and 5 respectively (see Proposition 7.17 when $\bar{\rho}$ is non-semisimple).

The strategy is to first arrive at an initial guess for the deformation ring, whose p -saturation is the deformation ring. However, since the equations are not explicit (although they are modulo p), it is a priori unclear whether our guessed ring is already p -flat. We then invoke global arguments in the form of the Serre weight conjectures (Theorem 7.8 and Proposition 7.17) that the candidate ring is in fact p -flat.

With respect to the notations of § 2.1, we have $f = 1$ and set $a \stackrel{\text{def}}{=} a_{s_0(1),0}$, $b \stackrel{\text{def}}{=} a_{s_0(2),0}$ and $c \stackrel{\text{def}}{=} a_{s_0(3),0}$.

8.1. The identity shape. The universal family of shape id is given by

$$A = \begin{pmatrix} c_{11} + c_{11}^*(v+p) & c_{12} & c_{13} \\ vc_{21} & c_{22} + c_{22}^*(v+p) & c_{23} \\ vc_{31} & vc_{32} & c_{33} + c_{33}^*(v+p) \end{pmatrix}$$

subject to the conditions that all 2 by 2 minors of

$$A|_{v=-p} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ -pc_{21} & c_{22} & c_{23} \\ -pc_{31} & -pc_{32} & c_{33} \end{pmatrix}$$

vanish and the determinant condition

$$c_{11}c_{22}^*c_{33}^* + c_{22}c_{33}^*c_{11}^* + c_{33}c_{11}^*c_{22}^* - c_{11}^*c_{23}c_{32} - c_{22}^*c_{13}c_{31} - c_{33}^*c_{12}c_{21} + c_{21}c_{13}c_{32} = 0.$$

We have

$$-A^\dagger|_{v=-p} = \begin{pmatrix} -pec_{11}^* & (a-b)c_{12} & (a-c)c_{13} \\ -p(e+b-a)c_{21} & -pec_{22}^* & (b-c)c_{23} \\ -p(e+c-a)c_{31} & -p(e+c-b)c_{32} & -pec_{33}^* \end{pmatrix}, \frac{\det(A)}{P(v)}A^{-1}|_{v=-p} = \begin{pmatrix} c_{22}^*c_{33} + c_{33}^*c_{22} - c_{23}c_{32} & c_{13}c_{32} - c_{33}^*c_{12} & -c_{22}^*c_{13} \\ -c_{33}^*c_{21} & c_{11}^*c_{33} + c_{33}^*c_{11} - c_{13}c_{31} & c_{13}c_{21} - c_{11}^*c_{23} \\ -pc_{21}c_{32} + pc_{22}^*c_{31} & pc_{11}^*c_{32} & c_{11}^*c_{22} + c_{22}^*c_{11} - c_{12}c_{21} \end{pmatrix}.$$

The leading term for the monodromy is given by $(A^\dagger P(v)^2 A^{-1})_{v=-p}$ (cf. Definition 5.5). Define

$$\begin{aligned}\text{Mon}_1 &= (e - a + c)c_{22}^*c_{33} + (e - a + b)c_{22}c_{33}^* - (e - a + c)c_{23}c_{32} + pec_{22}^*c_{33}^* \\ \text{Mon}_2 &= (a - b)c_{33}^*c_{11} + (e - b + c)c_{33}c_{11}^* - (a - b)c_{133}c_{31} + pec_{33}^*c_{11}^* \\ \text{Mon}_3 &= (b - c)c_{11}^*c_{22} + (a - c)c_{11}c_{22}^* - (b - c)c_{12}c_{21} + pec_{11}^*c_{22}^*.\end{aligned}$$

The monodromy equations are

$$c_{ij}(\text{Mon}_j + O(p^2)) = 0$$

for $1 \leq i, j \leq 3$, and $O(p^2)$ stands for an error term which is divisible by p^2 . This is because the determinant condition shows that the ij -th entry of $\frac{\det(A)}{P(v)}A^{-1}|_{v=-p}$ is a sum of terms which are divisible by some c_{kj} for $1 \leq k \leq 3$ (for example, $c_{22}^*c_{33} + c_{33}^*c_{22} - c_{23}c_{32} = -(c_{22}^*c_{33}^*c_{11} + c_{22}^*c_{13}c_{31} + c_{33}^*c_{12}c_{21} - c_{21}c_{13}c_{32})/c_{11}^*$), and the rank ≤ 1 condition allows a conversion from $c_{ij}c_{kl}$ to $c_{il}c_{kj}$, at possibly a cost of a p in the denominator. Recall the \mathcal{O} -algebra $R_{\overline{\mathfrak{M}}, \overline{p}}^{\tau, \overline{\beta}, \square}$ from Definition 5.10 (2). By (5.7) and Theorem 4.16, it is isomorphic to a power series ring of 2 variables over the potentially crystalline ring $R_{\overline{p}}^{\mu, \tau}$, and hence has relative dimension 14 over \mathcal{O} . Recall also (cf. (5.7)) that we have a surjection $\pi : R_{\overline{\mathfrak{M}}}^{\tau, \overline{\beta}, \square} \twoheadrightarrow R_{\overline{\mathfrak{M}}, \overline{p}}^{\tau, \overline{\beta}, \square}$. The following proposition refines Theorem 5.13 in the present situation:

Proposition 8.1. *The surjection π factors through the quotient of $R_{\overline{\mathfrak{M}}}^{\tau, \overline{\beta}, \square}$ by the relations*

$$(8.1) \quad \text{Mon}_i + O(p^2) = 0, \forall i \in \{1, 2, 3\}.$$

Proof. We already know that $(\text{Mon}_j + O(p^2))c_{ij} = 0$. The refinement will come from the fact that $R_{\overline{p}}^{\mu, \tau}$ classifies representations with Hodge-Tate weights *exactly* $(2, 1, 0)$ instead of just being in $[0, 2]$.

Since $R_{\overline{p}}^{\mu, \tau}$ is flat and $R_{\overline{p}}^{\mu, \tau}[\frac{1}{p}]$ is regular, it suffices to show that the equations (8.1) hold in the flat closure of each connected component of $R_{\overline{\mathfrak{M}}, \overline{p}}^{\tau, \overline{\beta}, \square}[\frac{1}{p}]$. Let R denote the flat closure of a connected component of $R_{\overline{\mathfrak{M}}, \overline{p}}^{\tau, \overline{\beta}, \square}[\frac{1}{p}]$ and assume that we have $\text{Mon}_1 + O(p^2) \neq 0$ in R . Since R is a domain, we conclude that $c_{i1} = 0$ in R . On the other hand, not all $c_{ij} = 0$ in R , so at least one of the equations $\text{Mon}_i + O(p^2) = 0$ for $i = 2, 3$ holds in R . One then checks that this implies that $c_{22}, c_{33} \in pR^\times$, and then R is a quotient of a ring of dimension < 14 . This is a contradiction, since R has dimension 14. \square

Corollary 8.2. *Let \tilde{R} be the quotient of $\mathbb{F}\llbracket c_{ij}, 1 \leq i, j, \leq 3 \rrbracket$ by the relations:*

$$\begin{aligned}c_{ii}c_{jj} &= 0, & \text{for } i \neq j; \\ c_{11}c_{23} &= 0; & c_{31}c_{22} = 0; & c_{12}c_{23} = c_{22}c_{13}; & c_{11}c_{32} = c_{12}c_{31}; & c_{21}c_{33} = c_{31}c_{23};\end{aligned}$$

$$\begin{aligned}
(e - a + c)c_{33} + (e - a + b)c_{22} - (e - a + c)c_{23}c_{32} &= 0; \\
(b - c)c_{22} + (a - c)c_{11} - (b - c)c_{12}c_{21} &= 0; \\
(a - b)c_{11} + (e - b + c)c_{33} - (a - b)c_{13}c_{31} &= 0; \\
c_{11} + c_{22} + c_{33} - c_{12}c_{21} - c_{13}c_{31} - c_{23}c_{32} + c_{21}c_{13}c_{32} &= 0.
\end{aligned}$$

Then the ring $R_{\overline{\mathfrak{M}}, \overline{\rho}}^{\tau, \beta, \square} / \varpi$ is a quotient of a power series ring over \tilde{R} .

Proof. This is a direct consequence of Proposition 8.1 and the observation that replacing c_{ij} by $\frac{c_{ij}}{c_{jj}^*}$ eliminates the c_{ii}^* from all the equations. \square

The following Proposition gives basic structural information about \tilde{R} :

Proposition 8.3. *The ring \tilde{R} is a reduced 3-dimensional Cohen-Macaulay ring. It has 6 minimal primes, and each irreducible component is formally smooth over \mathbb{F} . Thus $e(\tilde{R}) = 6$.*

Proof. Observe that the relations defining \tilde{R} are actually polynomials instead of genuine power series, so \tilde{R} can be viewed as a completion of a quotient of a polynomial ring by the ideal I generated by the relations above. We use some standard terminology from the theory of Gröbner basis [Eis95]. We pick the monomial order on $\mathbb{F}[c_{ij}] \stackrel{\text{def}}{=} \mathbb{F}[c_{ij}, 1 \leq i, j \leq 3]$ given by $c_{11} > c_{12} > c_{13} > c_{21} > c_{22} > c_{23} > c_{31} > c_{32} > c_{33}$ and write I for the ideal generated by the relations of Proposition 8.2. An easy but tedious calculation (using Buchberger's algorithm, for example) shows that the ideal I has the following Gröbner basis with our choice of monomial order:

$$\begin{aligned}
&c_{11} - c_{13}c_{31} + \frac{e - b + c}{a - b}c_{33}; \\
&c_{12}c_{21} - \frac{a - c}{b - c}c_{13}c_{31} - \frac{e - a + c}{e - a + b}c_{23}c_{32} + \left(\frac{(a - c)(e - b + c)}{(b - c)(a - b)} - \frac{e - a + c}{e - a + b} \right)c_{33}; \\
&c_{12}c_{23} - \frac{e - a + c}{e - a + b}c_{13}c_{33}; \\
&c_{12}c_{33}, c_{13}c_{21}c_{32} = \frac{a - c}{b - c}c_{13}c_{31} - c_{23}c_{32} + \frac{e}{b - c}c_{33}; \\
&c_{12}c_{23}c_{31} - \frac{e - b + c}{a - b}c_{23}c_{33}, c_{13}c_{31}c_{33} - \frac{e - b + c}{a - b}c_{33}^2; \\
&c_{21}c_{33} - c_{23}c_{31}, c_{22} - \frac{e - a + c}{e - a + b}c_{23}c_{32} + \frac{e - a + c}{e - a + b}c_{33}; \\
&c_{23}c_{31}c_{32} - c_{31}c_{33}, c_{23}c_{32}c_{33} - c_{33}^2;
\end{aligned}$$

In each of the above polynomials, the leading monomial is exactly the left-most term. Thus we see that the initial ideal $\text{in}(I)$ of I is generated by square-free monomials. This implies that I is radical:

Suppose $f^k \in I$, then $\text{in}(f)^k \in \text{in}(I)$, so $\text{in}(f) \in \text{in}(I)$. But then we can divide f by elements in I and get some $f' < f$ with $f'^k \in I$. Continuing this way, we see that $f \in I$.

Furthermore, as in [Eis95] §15.8, we can realize $\mathbb{F}[c_{ij}]/I$ as the fiber \mathcal{F}_t (for any $t \neq 0$) of a flat family \mathcal{F} over $\mathbb{F}[t]$, such that the fiber \mathcal{F}_0 is $\mathbb{F}[c_{ij}]/\text{in}(I)$. Since this quotient is Cohen-Macaulay by an explicit check, and the Cohen-Macaulay locus is open in \mathcal{F} , we conclude that \mathcal{F} is Cohen-Macaulay at some closed point of the form $c_{ij} = 0$, $t = t_0 \neq 0$. But then $(t - t_0)$ is a regular element in the localization of \mathcal{F} at this point, and hence the localization of $\mathbb{F}[c_{ij}]/I$ at the closed point $c_{ij} = 0$ is Cohen-Macaulay.

The computation of the irreducible components is left as an easy exercise to the reader. \square

Proposition 8.4. $R_{\overline{\mathfrak{M}}, \overline{\rho}}^{\tau, \overline{\beta}, \square} / \varpi$ is isomorphic to a power series ring over \tilde{R} .

Proof. We globalize $\overline{\rho}$ to a $\overline{\tau} : G_F \rightarrow \text{GL}_3(\mathbb{F})$ such that the following conditions hold:

- The assumptions of Theorem 7.8 are satisfied;
- $\overline{\tau}$ is unramified away from p ;
- p splits completely in F . Make a choice \tilde{v} above each place $v|p$ in F^+ ;
- For each $\tilde{v}|p$, there is an isomorphism $F_{\tilde{v}} \cong \mathbb{Q}_p$ and $\overline{\tau}|_{G_{F_{\tilde{v}}}} \cong \overline{\rho}$.

With this global setting, we can choose a weak minimal patching functor M_∞ . We now choose a tame type τ such that $\overline{\tau}|_{G_{F_{\tilde{v}}}}$ has a shape of length 4 with respect to τ_v for all but one place $v_0|p$, while $\overline{\tau}|_{G_{F_{v_0}}}$ has id shape with respect to τ_v . With this choice $\text{JH}(\sigma(\tau)) \cap W^2(\overline{\tau})$ consists of exactly 6 weights. For each global Serre weight W in this intersection we have $M_\infty(W) \neq 0$ by Theorem 7.8. Thus we have

$$e(M_\infty(\overline{\sigma}(\tau))) \geq 6.$$

On the other hand, by our choice of τ and the knowledge of Galois deformation rings for length 4 shapes (Corollary 5.14 and Table 6), $\overline{R}_\infty(\tau)$ is isomorphic to a power series ring over $R_{\overline{\rho}}^{\mu, \tau}$. It follows that $e(R_{\overline{\rho}}^{\mu, \tau} / \varpi) \geq 6$. The Proposition now follows from Lemma 8.5 below, the fact that $R_{\overline{\mathfrak{M}}, \overline{\rho}}^{\tau, \overline{\beta}, \square} / \varpi$ receives a surjection from a power series ring over \tilde{R} (Corollary 8.2) and has the same Hilbert-Samuel multiplicity as $R_{\overline{\rho}}^{\mu, \tau} / \varpi$. \square

Lemma 8.5. Suppose R, S are complete Noetherian local rings over \mathbb{F} , with $R \twoheadrightarrow S$. Assume that R is reduced, that R and S are equidimensional with $\dim R = \dim S$ and that $e(R) = e(S)$. Then $R \cong S$.

Proof. Let I be the ideal of definition for $R \twoheadrightarrow S$. Because $e(R) = e(S)$ and $\dim R = \dim S$, the support of I as an R -module does not contain any minimal primes, hence $I_{\mathfrak{p}} = 0$ for all minimal

primes \mathfrak{p} of R . But this implies that I is inside the intersection of all the minimal primes of R , hence $I = 0$ because R is reduced. \square

Corollary 8.6. *The ring $R_{\bar{\rho}}^{\mu, \tau}$ is normal, Cohen-Macaulay, and $R_{\bar{\rho}}^{\mu, \tau}[\frac{1}{p}]$ is a domain.*

Proof. By Propositions 8.3, 8.4 above, $R_{\bar{\rho}}^{\tau, \bar{\beta}, \square}$ is reduced and Cohen-Macaulay, hence $R_{\bar{\rho}}^{\mu, \tau}/\varpi$ inherits those properties by formal smoothness (cf. (5.7)). This implies Cohen-Macaulayness.

Since $R_{\bar{\rho}}^{\mu, \tau}[\frac{1}{p}]$ is regular, to show it is a domain it suffices to show it has no non-trivial idempotent. Suppose e is a non-trivial idempotent. Then there is a maximal $k \in \mathbb{Z}$ such that $\varpi^{-k}e \in R_{\bar{\rho}}^{\mu, \tau}$. By maximality and $e = e^2 \in \varpi^{-2k}R_{\bar{\rho}}^{\mu, \tau}$, we have $k \geq 2k$. On the other hand $(\varpi^{-k}e)^2 = \varpi^{-k}(\varpi^{-k}e)$ and $\varpi^{-k}e \neq 0 \pmod{\varpi}$, hence we must have $k = 0$ since $R_{\bar{\rho}}^{\mu, \tau}/\varpi$ is reduced. But then e is an idempotent of the local ring $R_{\bar{\rho}}^{\mu, \tau}$, hence it must be a trivial idempotent.

Finally, since $R_{\bar{\rho}}^{\mu, \tau}[\frac{1}{p}]$ is regular and $R_{\bar{\rho}}^{\mu, \tau}/\varpi$ is reduced, $R_{\bar{\rho}}^{\mu, \tau}$ satisfies conditions R_1 and S_2 , hence is normal. \square

8.2. The α shape. The universal family of shape α is given by

$$A = \begin{pmatrix} c_{11} & c_{12} + (v+p)c_{12}^* & c_{13} \\ c_{21}^*v & c_{22} + (v+p)c_{22}' & c_{23} \\ c_{31}v & c_{32}v & (c_{33} + (v+p)c_{33}^*) \end{pmatrix}$$

subject to the condition that all 2 by 2 minors of

$$A|_{v=-p} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ -pc_{21}^* & c_{22} & c_{23} \\ -pc_{31} & -pc_{32} & c_{33} \end{pmatrix}$$

vanish and the determinant condition

$$c_{11}c_{22}'c_{33}^* + c_{13}c_{21}^*c_{32} - c_{13}c_{22}'c_{31} - c_{12}c_{21}^*c_{33}^* + pc_{21}^*c_{12}^*c_{33}^* = 0.$$

We have

$$A_{v=-p}^\dagger = \begin{pmatrix} 0 & (a-b)c_{12} - pec_{12}^* & (a-c)c_{13} \\ -p(e-a+b)c_{21}^* & -epc_{22}' & (b-c)c_{23} \\ -p(e-a+c)c_{31} & -p(e-b+c)c_{32} & -pec_{33}^* \end{pmatrix}, \frac{\det(A)}{P(v)}A^{-1}|_{v=-p} = \begin{pmatrix} -c_{23}c_{32} + c_{22}c_{33}^* + c_{33}c_{22}' & c_{13}c_{32} - c_{12}c_{33}^* - c_{12}^*c_{33} & c_{12}^*c_{23} - c_{13}c_{22}' \\ pc_{33}^*c_{21}^* & -c_{13}c_{31} + c_{11}c_{33}^* & c_{21}^*c_{13} \\ -pc_{21}^*c_{32} + pc_{22}'c_{31} & pc_{31}c_{12}^* & -c_{21}^*c_{12} + c_{11}c_{22}' + pc_{12}^*c_{21}^* \end{pmatrix}$$

Define $\tilde{c}_{32} \stackrel{\text{def}}{=} c_{32} - \frac{c_{22}'c_{31}}{c_{21}^*}$. By looking at the $(1,1), (2,1), (3,3)$ entries of $(A^\dagger P(v)^2 A^{-1})_{v=-p}$ (the leading term for monodromy, cf. Definition 5.5) we get the following monodromy equations:

$$(a-b)c_{12}c_{33}^* - (a-c)c_{13}\tilde{c}_{32} = pec_{12}^*c_{33}^* + O(p^2)$$

$$(e-a+c)c_{23}\tilde{c}_{32} - (e-a+b)c_{22}c_{33}^* = epc_{22}'c_{33}^* + O(p^2)$$

$$(e-a+c)c_{31}c_{23}c_{12}^* - (e-a+c)c_{31}c_{13}c_{22}' + (e-b+c)c_{32}c_{13}c_{21}^* - ec_{12}c_{33}^*c_{21}^* + ec_{11}c_{22}'c_{33}^* + pec_{12}^*c_{21}^*c_{33}^* = O(p^2)$$

8.2.1. *The semisimple case.* Let us first consider the case when $\bar{c}'_{22} = 0$, that is $\bar{\rho}$ is semisimple.

Proposition 8.7. *Let \tilde{R} be the quotient of $\mathbb{F}[[c_{11}, c_{13}, c_{23}, c_{31}, \tilde{c}_{32}, c'_{22}]]$ by the relations:*

$$c_{11}c_{23} = 0; \quad c_{11}\tilde{c}_{32} = c_{13}c_{31}\tilde{c}_{32}; \quad c_{11}c'_{22} = \frac{b-c}{a-b}c_{13}\tilde{c}_{32}; \quad c_{13}c_{23}\tilde{c}_{32} = 0; \quad c_{23}c_{31}\tilde{c}_{32} = 0;$$

$$(a-b)c_{13}c_{31}c'_{22} + (c-b)c_{13}\tilde{c}_{32} + (e-a+c)c_{23}c_{31} = 0.$$

Then the ring $R_{\tilde{\mathfrak{M}}, \bar{\rho}}^{\tau, \bar{\beta}, \square} / \varpi$ is a quotient of a power series ring over \tilde{R} .

Proof. We need to check that the relations defining \tilde{R} are satisfied in $R_{\tilde{\mathfrak{M}}, \bar{\rho}}^{\tau, \bar{\beta}, \square} / \varpi$. Throughout the proof we work modulo ϖ . First, observe that replacing c_{i1} with $\frac{c_{i1}}{c_{21}^*}$, c_{i2} with $\frac{c_{i2}}{c_{12}^*}$ and c_{i3} with $\frac{c_{i3}}{c_{33}^*}$, we eliminate $c_{12}^*, c_{21}^*, c_{33}^*$ from all equations, so we can assume $c_{12}^* = c_{21}^* = c_{33}^* = 1$ in what follows. The monodromy equations mod ϖ solves c_{12}, c_{22} in terms of the remaining variables:

$$(8.2) \quad c_{12} = \frac{a-c}{a-b}c_{13}\tilde{c}_{32},$$

$$(8.3) \quad c_{22} = \frac{e-a+c}{e-a+b}c_{23}\tilde{c}_{32}.$$

The determinant condition thus gives

$$(8.4) \quad c_{11}c'_{22} = \frac{b-c}{a-b}c_{13}\tilde{c}_{32}.$$

From the relation $c_{11}c_{32} = c_{12}c_{31}$, using (8.2), the definition of \tilde{c}_{32} and (8.4), we obtain:

$$c_{11}\tilde{c}_{32} = c_{13}c_{31}\tilde{c}_{32}.$$

Multiplying (8.3) by c_{13} , (8.2) by c_{23} and using the relation $c_{12}c_{23} = c_{13}c_{22}$ we obtain:

$$c_{13}c_{23}\tilde{c}_{32} = 0.$$

Using $c_{22}c_{31} = 0$ and (8.3) we get

$$c_{23}c_{31}\tilde{c}_{32} = 0.$$

Finally, using the third monodromy equation and the previous relations we obtain

$$(a-b)c_{13}c_{31}c'_{22} + (c-b)c_{13}\tilde{c}_{32} + (e-a+c)c_{23}c_{31} = 0.$$

□

Proposition 8.8. *The ring \tilde{R} is a reduced 3-dimensional Cohen-Macaulay ring. It has 6 minimal primes, and each irreducible component is formally smooth over \mathbb{F} . Thus $e(\tilde{R}) = 6$. Furthermore, the minimal primes of \tilde{R} are exactly*

$$\begin{aligned} & (c_{11} - c_{13}c_{31}, c_{23}, (a - b)c_{31}c'_{22} + (c - b)\tilde{c}_{32}); & (c_{11}, (a - b)c_{13}c'_{22} + (e - a + c)c_{23}, \tilde{c}_{32}); \\ & (c_{11}, c_{13}, c_{23}); & (c_{11}, c_{13}, c_{31}); & (c_{11}, c_{31}, \tilde{c}_{32}); & (c_{23}, \tilde{c}_{32}, c'_{22}). \end{aligned}$$

Proof. The proof is very similar to the proof of Proposition 8.3, so we will only sketch it. The relation ideal I defining \tilde{R} consists of polynomials, and indeed form a Gröbner basis with respect to the monomial order $c_{11} > c_{13} > c_{21} > c_{23} > c_{31} > \tilde{c}_{32} > c'_{22}$. Since the initial ideal of I is generated by square-free monomials, we get reducedness. Cohen-Macaulayness follows as in Proposition 8.3. \square

The proof of the following results are exactly the same as the proofs for the id shape, so we will not repeat it.

Proposition 8.9. $R_{\overline{\mathfrak{M}}, \overline{\rho}}^{\tau, \overline{\beta}, \square} / \varpi$ is isomorphic to a power series ring over \tilde{R} .

Corollary 8.10. The ring $R_{\overline{\rho}}^{\mu, \tau}$ is normal, Cohen-Macaulay, and $R_{\overline{\mathfrak{M}}}^{\mu, \tau}[\frac{1}{p}]$ is a domain.

8.2.2. The non-semisimple case. Finally, we handle the case where $\overline{\rho}$ is non-semisimple. Then we have that c'_{22} is a unit instead of a topologically nilpotent element. The relations of Propositions 8.7, 8.8 continue to hold, the only difference is that \tilde{R} is not a quotient of $\mathbb{F}[[c_{11}, c_{13}, c_{21}, c_{23}, c_{31}, \tilde{c}_{32}, c'_{22}]]$, but rather a quotient of $\mathbb{F}[[c_{11}, c_{13}, c_{21}, c_{23}, c_{31}, \tilde{c}_{32}, c'_{22} - [\tilde{c}_{22}]]]$. The effect of c'_{22} being a unit is that \tilde{R} only has 5 minimal primes instead of 6 (the minimal prime $(c_{23}, c_{32}, c'_{22})$ is no longer present).

This completes the proof of Theorem 7.7.

9. APPENDIX: TABLES

TABLE 1. **The $(2, 1, 0)$ -admissible elements**

Length 4	$\alpha\beta\alpha\gamma = t_{(2,1,0)}, \quad \beta\gamma\alpha\gamma = t_{(1,2,0)}, \quad \beta\gamma\beta\alpha = t_{(0,2,1)},$ $\gamma\alpha\beta\alpha = t_{(0,1,2)}, \quad \alpha\gamma\alpha\beta = t_{(1,0,2)}, \quad \alpha\beta\gamma\beta = t_{(2,0,1)}$
Length 3 (ordinary)	$\gamma\alpha\beta, \quad \alpha\gamma\beta, \quad \alpha\beta\gamma \quad \beta\alpha\gamma, \quad \beta\gamma\alpha, \quad \gamma\beta\alpha$
Length 3 (shadow)	$\gamma\alpha\gamma, \quad \alpha\beta\alpha, \quad \beta\gamma\beta$
Length 2	$\gamma\alpha, \quad \alpha\gamma, \quad \beta\alpha, \quad \alpha\beta, \quad \beta\gamma, \quad \gamma\beta$
Length 1	$\alpha, \quad \beta, \quad \gamma$
Length 0	id

There are 25 different $(2, 1, 0)$ -admissible elements. For simplicity, we label them by the corresponding element in the affine Weyl group of SL_3 , e.g. $\alpha\beta\alpha\gamma$ corresponds to $v(\alpha\beta\alpha\gamma)$ in \widetilde{W} . If $(x, y, z) \in X_*(T) \cong \mathbb{Z}^3$ is a cocharacter, we write $t_{(x,y,z)}$ for the image of translation by (x, y, z) in \widetilde{W} .

TABLE 2. **Inertial local Langlands**

τ	$\sigma(\tau)$
$\omega_f^{-\mathbf{a}_1^{(0)}} \oplus \omega_f^{-\mathbf{a}_2^{(0)}} \oplus \omega_f^{-\mathbf{a}_3^{(0)}}$	$\mathrm{Ind}_{\mathrm{B}(k)}^{\mathrm{GL}_3(k)}(\tilde{\omega}_f^{\mathbf{a}_1^{(0)}} \otimes \tilde{\omega}_f^{\mathbf{a}_2^{(0)}} \otimes \tilde{\omega}_f^{\mathbf{a}_3^{(0)}})$
$\omega_f^{-\mathbf{a}_1^{(0)}} \oplus \omega_{2f}^{-\mathbf{a}_2^{(0)} - p^f \mathbf{a}_3^{(0)}} \oplus \omega_{2f}^{-\mathbf{a}_3^{(0)} - p^f \mathbf{a}_2^{(0)}}$	$\mathrm{Ind}_{P_2(k)}^{\mathrm{GL}_3(k)} \tilde{\omega}_f^{\mathbf{a}_1^{(0)}} \otimes \Theta(\tilde{\omega}_{2f}^{\mathbf{a}_2^{(0)} + p^f \mathbf{a}_3^{(0)}})$
$\omega_{3f}^{-\mathbf{a}_1^{(0)} - p^f \mathbf{a}_2^{(0)} - p^{2f} \mathbf{a}_3^{(0)}} \oplus \omega_{3f}^{-\mathbf{a}_2^{(0)} - p^f \mathbf{a}_3^{(0)} - p^{2f} \mathbf{a}_1^{(0)}} \oplus \omega_{3f}^{-\mathbf{a}_3^{(0)} - p^f \mathbf{a}_1^{(0)} - p^{2f} \mathbf{a}_2^{(0)}}$	$\Theta(\tilde{\omega}_{3f}^{\mathbf{a}_1^{(0)} + p^f \mathbf{a}_2^{(0)} + p^{2f} \mathbf{a}_3^{(0)}})$

In the table above, we set $P_2 \stackrel{\mathrm{def}}{=} \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$ and write $\Theta(\psi)$ for the cuspidal representation of $\mathrm{GL}_r(k)$ associated to a k' -primitive character $\psi : k' \rightarrow E^\times$ as in [Her09, Lemma 4.4].

TABLE 3. Shapes of Kisin modules over \mathbb{F}

\tilde{w}_j	$\overline{A}_{\tilde{w}_j}^{(j)}$	\tilde{w}_j	$\overline{A}_{\tilde{w}_j}^{(j)}$
$\alpha\beta\alpha\gamma$	$\begin{pmatrix} v^2\overline{c}_{11}^* & 0 & 0 \\ v^2\overline{c}_{21} & v\overline{c}_{22}^* & 0 \\ \overline{c}_{31}v + \overline{c}'_{31}v^2 & v\overline{c}_{32} & \overline{c}_{33}^* \end{pmatrix}$	$\beta\gamma\alpha\gamma$	$\begin{pmatrix} v\overline{c}_{11}^* & v\overline{c}_{12} & 0 \\ 0 & v^2\overline{c}_{22}^* & 0 \\ v\overline{c}_{31} & \overline{c}_{32}v + \overline{c}'_{32}v^2 & \overline{c}_{33}^* \end{pmatrix}$
$\beta\alpha\gamma$	$\begin{pmatrix} 0 & v\overline{c}_{12}^* & 0 \\ v^2\overline{c}_{21}^* & 0 & 0 \\ v\overline{c}_{31} + v^2\overline{c}'_{31} & v\overline{c}_{32} & \overline{c}_{33}^* \end{pmatrix}$	$\alpha\beta\gamma$	$\begin{pmatrix} v^2\overline{c}_{11}^* & 0 & 0 \\ v\overline{c}_{21} + v^2\overline{c}'_{21} & 0 & \overline{c}_{23}^* \\ v^2\overline{c}_{31} & v\overline{c}_{32}^* & 0 \end{pmatrix}$
$\alpha\beta\alpha$	$\begin{pmatrix} 0 & 0 & v\overline{c}_{13}^* \\ 0 & v\overline{c}_{22}^* & v\overline{c}_{23} \\ v\overline{c}_{31}^* & v\overline{c}_{32} & v\overline{c}_{33} \end{pmatrix}$		
$\alpha\beta$	$\begin{pmatrix} 0 & 0 & v\overline{c}_{13}^* \\ v\overline{c}_{21}^* & 0 & v\overline{c}_{23} \\ 0 & v\overline{c}_{32}^* & v\overline{c}_{33} \end{pmatrix}$	$\beta\alpha$	$\begin{pmatrix} 0 & v\overline{c}_{12}^* & 0 \\ 0 & 0 & v\overline{c}_{23}^* \\ v\overline{c}_{31}^* & v\overline{c}_{32} & v\overline{c}_{33} \end{pmatrix}$
α	$\begin{pmatrix} 0 & v\overline{c}_{12}^* & 0 \\ v\overline{c}_{12}^* & v\overline{c}_{22} & 0 \\ 0 & 0 & v\overline{c}_{33}^* \end{pmatrix}$	id	$\begin{pmatrix} v\overline{c}_{11}^* & 0 & 0 \\ 0 & v\overline{c}_{22}^* & 0 \\ 0 & 0 & v\overline{c}_{33}^* \end{pmatrix}$

In the above, we have $\overline{c}_{ik}, \overline{c}'_{ik} \in \overline{\mathbb{F}}$ and $\overline{c}_{ik}^* \in \overline{\mathbb{F}}^\times$.

TABLE 4. Deforming $\overline{\mathfrak{M}}$ by shape (without monodromy)

\tilde{w}_j	$\overline{A}_{\tilde{w}_j}^{(j)}$	$\deg(\tilde{A}_{\tilde{w}_j}^{(j)})$	With height/det conditions
$\alpha\beta\alpha\gamma$	$\begin{pmatrix} v^2\overline{c}_{11}^* & 0 & 0 \\ v^2\overline{c}_{21} & v\overline{c}_{22}^* & 0 \\ v\overline{c}_{31} + v^2\overline{c}'_{31} & v\overline{c}_{32} & \overline{c}_{33}^* \end{pmatrix}$	$\begin{pmatrix} 2^* & \leq 0 & -\infty \\ v(\leq 1) & 1^* & -\infty \\ v(\leq 1) & v(\leq 0) & 0^* \end{pmatrix}$	$\begin{pmatrix} (v+p)^2c_{11}^* & 0 & 0 \\ v(v+p)c_{21} & (v+p)c_{22}^* & 0 \\ vc_{31} + v^2c'_{31} & vc_{32} & c_{33}^* \end{pmatrix}$
$\beta\gamma\alpha\gamma$	$\begin{pmatrix} v\overline{c}_{11}^* & v\overline{c}_{12} & 0 \\ 0 & v^2\overline{c}_{22}^* & 0 \\ v\overline{c}_{31} & v\overline{c}_{32} + v^2\overline{c}'_{32} & \overline{c}_{33}^* \end{pmatrix}$	$\begin{pmatrix} 1^* & \leq 1 & -\infty \\ v(\leq 0) & 2^* & -\infty \\ v(\leq 0) & v(\leq 1) & 0^* \end{pmatrix}$	$\begin{pmatrix} (v+p)c_{11}^* & (v+p)c_{12} & 0 \\ 0 & (v+p)^2c_{22}^* & 0 \\ vc_{31} & vc_{32} + v^2c'_{32} & c_{33}^* \end{pmatrix}$
$\beta\alpha\gamma$	$\begin{pmatrix} 0 & v\overline{c}_{12}^* & 0 \\ v^2\overline{c}_{21}^* & 0 & 0 \\ v\overline{c}_{31} + v^2\overline{c}'_{31} & v\overline{c}_{32} & \overline{c}_{33}^* \end{pmatrix}$	$\begin{pmatrix} \leq 1 & 1^* & -\infty \\ v(1^*) & \leq 1 & -\infty \\ v(\leq 1) & v(\leq 0) & 0^* \end{pmatrix}$	$\begin{pmatrix} (v+p)c_{11} & (v+p)c_{12}^* & 0 \\ v(v+p)c_{21}^* & (v+p)c_{22} & 0 \\ vc_{31} + v^2c'_{31} & vc_{32} & c_{33}^* \end{pmatrix}$ $c_{11}c_{22} = -pc_{12}^*c_{21}^*$
$\alpha\beta\gamma$	$\begin{pmatrix} v^2\overline{c}_{11}^* & 0 & 0 \\ v\overline{c}_{21} + v^2\overline{c}'_{21} & 0 & \overline{c}_{23}^* \\ v^2\overline{c}'_{31} & v\overline{c}_{32}^* & 0 \end{pmatrix}$	$\begin{pmatrix} 2^* & \leq 0 & -\infty \\ v(\leq 1) & \leq 0 & 0^* \\ v(\leq 1) & v(0^*) & \leq 0 \end{pmatrix}$	$\begin{pmatrix} (v+p)^2c_{11}^* & 0 & 0 \\ v(c_{21} + (v+p)c'_{21}) & c_{22} & c_{23}^* \\ v(c_{21}c_{33}(c_{23}^*)^{-1} + (v+p)c'_{31}) & vc_{32}^* & c_{33} \end{pmatrix}$ $c_{22}c_{33} = -pc_{32}^*c_{23}^*$
$\alpha\beta\alpha$	$\begin{pmatrix} 0 & 0 & v\overline{c}_{13}^* \\ 0 & v\overline{c}_{22}^* & v\overline{c}_{23} \\ v\overline{c}_{31}^* & v\overline{c}_{32} & v\overline{c}'_{33} \end{pmatrix}$	$\begin{pmatrix} \leq 0 & \leq 0 & 1^* \\ -\infty & 1^* & \leq 1 \\ v(0^*) & v(\leq 0) & \leq 1 \end{pmatrix}$	$\begin{pmatrix} c_{11} & c_{11}c_{32}(c_{31}^*)^{-1} & c_{13} + (v+p)c_{13}^* \\ 0 & (v+p)c_{22}^* & (v+p)c_{23} \\ vc_{31}^* & vc_{32} & c_{33} + (v+p)c'_{33} \end{pmatrix}$ $c_{11}c_{33} = -pc_{13}c_{31}^*$ $c_{11}c'_{33} - c_{13}c_{31}^* + pc_{13}^*c_{31}^* = 0$

$\alpha\beta$	$\begin{pmatrix} 0 & 0 & v\overline{c}_{13}^* \\ v\overline{c}_{21}^* & 0 & v\overline{c}_{23}^* \\ 0 & v\overline{c}_{32}^* & v\overline{c}_{33}^* \end{pmatrix}$	$\begin{pmatrix} \leq 0 & \leq 0 & 1^* \\ v(0^*) & \leq 0 & \leq 1 \\ v(\leq 0) & v(0^*) & \leq 1 \end{pmatrix}$	$\begin{pmatrix} c_{31}c_{12}(c_{32}^*)^{-1} & c_{12} & c_{13} + (v+p)c_{13}^* \\ vc_{21}^* & c_{22} & c_{23} + (v+p)c_{23}^* \\ vc_{31} & vc_{32}^* & (c_{31}c_{23}(c_{21}^*)^{-1} + (v+p)c_{33}^*) \end{pmatrix}$ $c_{22}c_{31} = -pc_{21}^*c_{32}^*$ $c_{12}c_{23} = c_{22}c_{13}$ $c_{21}^*c_{32}^*c_{13} - pc_{21}^*c_{32}^*c_{13}^* - c_{33}^*c_{21}^*c_{12} = 0$
$\beta\alpha$	$\begin{pmatrix} 0 & v\overline{c}_{12}^* & 0 \\ 0 & 0 & v\overline{c}_{23}^* \\ v\overline{c}_{31}^* & v\overline{c}_{32}^* & v\overline{c}_{33}^* \end{pmatrix}$	$\begin{pmatrix} \leq 0 & 1^* & \leq 0 \\ -\infty & \leq 1 & 1^* \\ v(0^*) & v(\leq 0) & \leq 1 \end{pmatrix}$	$\begin{pmatrix} c_{11} & ((c_{31}^*)^{-1}c_{11}c_{32} + (v+p)c_{12}^*) & c_{13} \\ 0 & (v+p)c_{22}^* & (v+p)c_{23}^* \\ c_{31}^*v & c_{32}v & c_{33} + (v+p)c_{33}^* \end{pmatrix}$ $c_{11}c_{33} = -pc_{31}^*c_{13}$ $c_{22}^*(c_{11}c_{33}^* - c_{13}c_{31}^*) = -pc_{23}^*c_{12}^*c_{31}^*$
α	$\begin{pmatrix} 0 & v\overline{c}_{12}^* & 0 \\ v\overline{c}_{12}^* & v\overline{c}_{22}^* & 0 \\ 0 & 0 & v\overline{c}_{33}^* \end{pmatrix}$	$\begin{pmatrix} \leq 0 & 1^* & \leq 0 \\ v(0^*) & \leq 1 & \leq 0 \\ v(\leq 0) & v(\leq 0) & 1^* \end{pmatrix}$	$\begin{pmatrix} c_{11} & c_{12} + (v+p)c_{12}^* & c_{13} \\ c_{21}^*v & c_{22} + (v+p)c_{22}^* & c_{23} \\ c_{31}v & c_{32}v & ((c_{21}^*)^{-1}c_{31}c_{23} + (v+p)c_{33}^*) \end{pmatrix}$ $c_{31}c_{22} = -pc_{21}^*c_{32}, \quad c_{11}c_{23} = -pc_{21}^*c_{13}, \quad c_{11}c_{22} = -pc_{21}^*c_{12},$ $c_{11}c_{22}^*c_{33}^* + c_{13}c_{21}^*c_{32} - c_{13}c_{22}^*c_{31} - c_{12}c_{21}^*c_{33}^* + pc_{21}^*c_{12}^*c_{33}^* = 0$
id	$\begin{pmatrix} v\overline{c}_{11}^* & 0 & 0 \\ 0 & v\overline{c}_{22}^* & 0 \\ 0 & 0 & v\overline{c}_{33}^* \end{pmatrix}$	$\begin{pmatrix} 1^* & \leq 0 & \leq 0 \\ v(\leq 0) & 1^* & \leq 0 \\ v(\leq 0) & v(\leq 0) & 1^* \end{pmatrix}$	$\begin{pmatrix} c_{11} + c_{11}^*(v+p) & c_{12} & c_{13} \\ vc_{21} & c_{22} + c_{22}^*(v+p) & c_{23} \\ vc_{31} & vc_{32} & c_{33} + c_{33}^*(v+p) \end{pmatrix}$ $c_{11}c_{22} = -pc_{21}c_{12}, \quad c_{11}c_{23} = -pc_{21}c_{13}, \quad c_{11}c_{33} = -pc_{31}c_{13}, \quad c_{12}c_{33} = -pc_{32}c_{13}$ $c_{11}^*(c_{22}c_{33} + pc_{22}^*c_{33}^*) + c_{22}^*(c_{11}c_{33} + pc_{11}^*c_{33}^*) + c_{33}^*(c_{11}c_{22} + pc_{11}^*c_{22}^*) = 0$

Explanation of the table: $\deg(\tilde{A}_{\tilde{w}_j}^{(j)})$ means the degree of the polynomial in each entry. We write k^* to indicate an entry polynomial of degree k whose leading coefficient is a unit. We use c^* to indicate an entry which is a unit in R . Each entry is also subject to the condition that the reduction modulo m_R gives $\overline{A}_{\tilde{w}_j}^{(j)}$. The third column is further explained in Remark 4.7.

TABLE 5. Monodromy equations

\tilde{w}_j	$R_{\tilde{w}_j}^{\text{expl}}$	Leading term
$\alpha\beta\alpha\gamma$	$\mathcal{O}[[x_{11}^*, x_{22}^*, x_{33}^*, x_{21}, x_{31}, x'_{31}, x_{32}]]$	c_{31}
$\beta\gamma\alpha\gamma$	$\mathcal{O}[[x_{11}^*, x_{22}^*, x_{33}^*, x_{12}, x_{31}, x_{32}, x'_{32}]]$	$(e - b + c)c_{32}c_{11}^* - (e - a + c)c_{12}c_{31}$
$\beta\alpha\gamma$	$\mathcal{O}[[x_{11}, x_{12}^*, x_{21}^*, x_{33}^*, x_{22}, x_{31}, x'_{31}, x_{32}]]$ $c_{11}c_{22} = -pc_{12}^*c_{21}^*$	$c_{33}^*((e - (a - c))c_{12}^*c_{31} - p(2e - (a - c))c_{12}^*c'_{31} - (e - (b - c))c_{32}c_{11})$
$\alpha\beta\gamma$	$\mathcal{O}[[x_{11}^*, x_{21}, x'_{21}, x_{22}, x_{23}^*, x'_{31}, x_{32}^*, x_{33}]]$ $c_{22}c_{33} = -pc_{32}^*c_{23}^*$	$c_{23}^*((e - (a - c))c_{32}^*c_{21} + (b - c)c_{22}c_{31} - p(e - (b - c))c_{32}^*c'_{21})$
$\alpha\beta\alpha$	$\mathcal{O}[[x_{11}, x_{32}, x_{23}, x_{13}, x_{33}, x'_{33}, x_{31}^*, x_{22}^*, x_{13}^*]]$ $c_{11}c_{33} = -pc_{13}c_{31}^*$ $c_{11}c'_{33} - c_{13}c_{31}^* + pc_{13}^*c_{31} = 0$	$c_{11}((a - b)c_{23}c_{32} - (a - c)c_{22}^*c'_{33}) - p(e - a + c)c_{31}^*c_{22}^*c_{13}^*$
$\alpha\beta$	$\mathcal{O}[[x_{31}, x_{22}, x_{12}, x_{13}, x_{23}, x'_{23}, x'_{33}, x_{21}^*, x_{13}^*, x_{32}^*]]$ $c_{22}c_{31} = -pc_{21}^*c_{32}^*$ $c_{12}c_{23} = c_{22}c_{13}$ $c_{32}^*c_{13} - pc_{32}^*c_{13}^* - c'_{33}c_{12} = 0$	$c_{12}((b - c)c'_{33}c_{21}^* + (a - b)c_{31}c'_{23}) + p(e - a + c)c_{21}^*c_{13}^*c_{32}^*$
$\beta\alpha$	$\mathcal{O}[[x_{11}, x'_{22}, x_{32}, x_{13}, x_{33}, x'_{33}, x_{31}^*, x_{12}^*, x_{23}^*]]$ $c_{11}c_{33} = -pc_{31}^*c_{13}$ $c'_{22}(c_{11}c'_{33} - c_{13}c_{31}^*) = -pc_{23}^*c_{12}^*c_{31}^*$	$c_{11}((a - c)c'_{22}c'_{33} + (a - b)c_{23}^*c_{32}) - p(e - a + c)c_{31}^*c_{12}^*c_{13}^*$
α	$\mathcal{O}[[x_{11}, x_{12}, x_{12}^*, x_{13}, x_{21}^*, x_{22}, x'_{22}, x_{23}, x_{31}, x_{32}, x_{33}^*]]$ $c_{31}c_{22} = -pc_{21}^*c_{32}, c_{11}c_{23} = -pc_{21}^*c_{13}, c_{11}c_{22} = -pc_{21}^*c_{12},$ $c_{11}c'_{22}c_{33}^* + c_{13}c_{21}^*c_{32} - c_{13}c'_{22}c_{31} - c_{12}c_{21}^*c_{33}^* + pc_{21}^*c_{12}^*c_{33}^* = 0$	$(e - a + c)(c_{23}c_{32}c_{21}^* - c'_{22}c_{23}c_{31}) - (e - a + b)c_{22}c_{33}^*c_{21}^* - epc'_{22}c_{33}^*c_{21}^*$

In the table, we take $a \stackrel{\text{def}}{=} a_{s_j(1)}^{(j)}$, $b \stackrel{\text{def}}{=} a_{s_j(2)}^{(j)}$, $c \stackrel{\text{def}}{=} a_{s_j(3)}^{(j)}$ and the variables $x_{ij}^\bullet \stackrel{\text{def}}{=} c_{ij}^\bullet - [\bar{c}_{ij}^\bullet]$ where $\bullet = *, \prime$ or $\bullet \in \emptyset$ and $[\cdot]$ is the Teichmüller lift.

TABLE 6. Deformation rings with monodromy

\tilde{w}_j	Condition on $\overline{\mathfrak{M}}$	$R_{\overline{\mathfrak{M}}, \tilde{w}_j}^{\text{expl}, \nabla}$
$\alpha\beta\alpha\gamma$	$\bar{c}_{31} = 0$	$\mathcal{O}[[x_{11}^*, x_{22}^*, x_{33}^*, x_{21}, x'_{31}, x_{32}]]$
$\beta\gamma\alpha\gamma$	$(e - b + c)\bar{c}_{32}\bar{c}_{11}^* = (e - a + c)\bar{c}_{12}\bar{c}_{31}$	$\mathcal{O}[[x_{11}^*, x_{22}^*, x_{33}^*, x_{12}, x_{31}, x'_{32}]]$
$\beta\alpha\gamma$	$\bar{c}_{31} = 0$	$\mathcal{O}[[y_{11}, y_{22}, x_{12}^*, x_{21}^*, x_{33}^*, x'_{31}, x_{32}]]/(y_{11}y_{22} - p)$
$\alpha\beta\gamma$	$\bar{c}_{21} = 0$	$\mathcal{O}[[y_{22}, y_{33}, x_{11}^*, x'_{21}, x_{23}^*, x'_{31}, x_{32}^*]]/(y_{22}y_{33} - p)$
$\alpha\beta\alpha$	$(a - b)\bar{c}_{23}\bar{c}_{32} - (a - c)\bar{c}_{22}^*\bar{c}'_{33} \neq 0$	$\mathcal{O}[[x_{32}, x_{23}, x'_{33}, x_{31}^*, x_{22}^*, x_{13}^*]]$
$\alpha\beta\alpha$	$(a - b)\bar{c}_{23}\bar{c}_{32} - (a - c)\bar{c}_{22}^*\bar{c}'_{33} = 0$	$\mathcal{O}[[x_{11}, x_{32}, x_{23}, y'_{33}, x_{31}^*, x_{22}^*, x_{13}^*]]/(x_{11}y'_{33} - p)$
$\alpha\beta$	$\bar{c}'_{33} \neq 0$	$\mathcal{O}[[y_{31}, x_{22}, x'_{23}, x'_{33}, x_{21}^*, x_{13}^*, x_{32}^*]]/(y_{31}x_{22} - p)$
$\alpha\beta$	$\bar{c}'_{33} = 0$	$\mathcal{O}[[y_{31}, x_{22}, x_{12}, x'_{23}, y'_{33}, x_{21}^*, x_{13}^*, x_{32}^*]]/(y_{31}x_{22} - p, x_{12}y'_{33} - p)$
$\beta\alpha$	$\bar{c}_{32} \neq 0$	$\mathcal{O}[[x'_{22}, y_{13}, x_{32}, x'_{33}, x_{31}^*, x_{12}^*, x_{23}^*]]/(x'_{22}y_{13} - p)$
$\beta\alpha$	$\bar{c}_{32} = 0$	$\mathcal{O}[[x_{11}, x'_{22}, y_{32}, y_{13}, x'_{33}, x_{31}^*, x_{12}^*, x_{23}^*]]/(x'_{22}y_{13} - p, x_{11}y_{32} - p)$

The condition imposed by monodromy on the coefficients of $\text{Mat}_{\bar{\beta}}(\phi_{\overline{\mathfrak{M}}, s_{j+1}(3)}^{(j)})$, according to the shape of $\overline{\mathfrak{M}}$ (cf. Table 3). Note that the table above covers all the shapes of length ≥ 2 ; the shapes of length ≤ 1 are more delicate and treated in detail in §8.

TABLE 7. Types with Weyl intersection

$\bar{\rho}$	$\sigma(\tau)$	$\mathrm{JH}(\sigma(\tau)) \cap W^?(\bar{\rho})$	Shape $\mathbf{w}(\bar{\rho}, \tau)$
$(\omega^a \oplus \omega^b \oplus \omega^c) \otimes \omega$	$\mathrm{Ind}_{P_1(\mathbb{F}_p)}^{\mathrm{GL}_3(\mathbb{F}_p)}(\tilde{\omega}^b \otimes \Theta(\tilde{\omega}_2^{c+pa}))$	$W(c+p-1, b, a-p+1)$	$\alpha\beta\alpha$
	$\mathrm{Ind}_{P_1(\mathbb{F}_p)}^{\mathrm{GL}_3(\mathbb{F}_p)}(\tilde{\omega}^c \otimes \Theta(\tilde{\omega}_2^{(a+1)+p(b-1)}))$	$W(a, c, b-p+1)$	$\beta\gamma\beta$
	$\mathrm{Ind}_{P_1(\mathbb{F}_p)}^{\mathrm{GL}_3(\mathbb{F}_p)}(\tilde{\omega}^a \otimes \Theta(\tilde{\omega}_2^{(b+1)+p(c-1)}))$	$W(b+p-1, a, c)$	$\gamma\alpha\gamma$
$(\omega^a \oplus \mathrm{Ind}_{G_{\mathbb{Q}_p^2}}^{G_{\mathbb{Q}_p}} \omega_2^{b+pc}) \otimes \omega$	$\Theta(\tilde{\omega}_3^{a+pb+p^2c})$	$W(c+p-1, b, a-p+1)$	$\alpha\beta\alpha$
	$\Theta(\tilde{\omega}_3^{(c-1)+pb+p^2(a+1)})$	$W(a, c, b-p+1)$	$\beta\gamma\beta$
	$\mathrm{Ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_3(\mathbb{F}_p)}(\omega^a \otimes \omega^{b-1} \otimes \omega^{c+1})$	$W(b+p-2, a, c+1)$	$\gamma\alpha\gamma$
$(\mathrm{Ind}_{G_{\mathbb{Q}_p^3}}^{G_{\mathbb{Q}_p}} \omega_3^{a+pb+p^2c}) \otimes \omega$	$\mathrm{Ind}_{P_1(\mathbb{F}_p)}^{\mathrm{GL}_3(\mathbb{F}_p)}(\omega^a \otimes \Theta(\tilde{\omega}_2^{b+pc}))$	$W(c+p-1, b, a-p+1)$	$\alpha\beta\alpha$
	$\mathrm{Ind}_{P_1(\mathbb{F}_p)}^{\mathrm{GL}_3(\mathbb{F}_p)}(\omega^{c+1} \otimes \Theta(\tilde{\omega}_2^{(a-1)+pb}))$	$W(b+p-1, a-1, c+1)$	$\gamma\alpha\gamma$
	$\mathrm{Ind}_{P_1(\mathbb{F}_p)}^{\mathrm{GL}_3(\mathbb{F}_p)}(\omega^{b+1} \otimes \Theta(\tilde{\omega}_2^{(c-1)+pa}))$	$W(a-1, c, b-p+2)$	$\beta\gamma\beta$
$(\omega^b \oplus \mathrm{Ind}_{G_{\mathbb{Q}_p^2}}^{G_{\mathbb{Q}_p}} \omega_2^{a+pc}) \otimes \omega$	$\mathrm{Ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_3(\mathbb{F}_p)}(\omega^a \otimes \omega^b \otimes \omega^c)$	$W(c+p-1, b, a-p+1)$	$\alpha\beta\alpha$
	$\Theta(\tilde{\omega}_3^{c+p(b+1)+p^2(a-1)})$	$W(b+p-1, a-1, c+1)$	$\gamma\alpha\gamma$
	$\Theta(\tilde{\omega}_3^{a+p(b-1)+p^2(c+1)})$	$W(a-1, c+1, b-p+1)$	$\beta\gamma\beta$

In the above, $(a, b, c) \in \mathbb{Z}^3$ is a p -restricted triple verifying $a - c < p - 1$. For $\lambda = (x, y, z) \in \mathbb{Z}^3$ dominant, we write $W(x, y, z)$ for the \mathbb{F}_p -specialization of the algebraic induction $\mathrm{Ind}_{B_3}^{\mathrm{GL}_3}(\omega_0 \cdot \lambda)_{/\mathbb{F}_p}$.

TABLE 8. Shapes and modular weights for type $\tau = \omega^{-a} \oplus \omega^{-b} \oplus \omega^{-c}$.

\tilde{w}	$\overline{Y}_{\tilde{w}}^{\mu, \tau}$	Special loci	$\overline{\rho}$	$\mathrm{JH}(\sigma(\tau)) \cap W^?(\overline{\rho})$
$\alpha\beta\alpha\gamma$	$\begin{pmatrix} v^2\overline{c}_{11}^* & 0 & 0 \\ v^2\overline{c}_{21} & v\overline{c}_{22}^* & 0 \\ \overline{c}_{31}v + \overline{c}_{31}'v^2 & v\overline{c}_{32} & \overline{c}_{33}^* \end{pmatrix}$	Monodromy locus: $\overline{c}_{31} = 0$.	$\begin{pmatrix} \omega^{a+2} & * & * \\ 0 & \omega^{b+1} & * \\ 0 & 0 & \omega^c \end{pmatrix}$	$F(a, b, c)$
$\beta\gamma\alpha\gamma$	$\begin{pmatrix} v\overline{c}_{11}^* & v\overline{c}_{12} & 0 \\ 0 & v^2\overline{c}_{22}^* & 0 \\ v\overline{c}_{31} & \overline{c}_{32}v + \overline{c}_{32}'v^2 & \overline{c}_{33}^* \end{pmatrix}$	Monodromy locus: $(b - c + 1)\overline{c}_{11}^*\overline{c}_{32} = (a - c + 1)\overline{c}_{12}\overline{c}_{31}$.	$\begin{pmatrix} \omega^{b+2} & * & * \\ 0 & \omega^{a+1} & * \\ 0 & 0 & \omega^c \end{pmatrix}$	$F(b + p - 1, a, c)$
$\beta\alpha\gamma$	$\begin{pmatrix} 0 & v & 0 \\ v^2\overline{c}_{21}^* & 0 & 0 \\ v\overline{c}_{31} + v^2\overline{c}_{31}' & v\overline{c}_{32} & \overline{c}_{33}^* \end{pmatrix}$	Monodromy locus: $\overline{c}_{31} = 0$. Other special loci: $\overline{c}_{31}' = 0$ $\overline{c}_{32} = 0$	$\begin{pmatrix} \mathrm{ind}_{G_{\mathbb{Q}_p}}^{G_{\mathbb{Q}_p}} \omega_2^{(a+2)+p(b+1)} & * \\ 0 & \omega^c \end{pmatrix}$	$F(a, b, c), F(b + p - 1, a, c)$
$\alpha\beta\gamma$	$\begin{pmatrix} v^2\overline{c}_{11}^* & 0 & 0 \\ v\overline{c}_{21} + v^2\overline{c}_{21}' & 0 & 1 \\ v^2\overline{c}_{31} & v\overline{c}_{32}^* & 0 \end{pmatrix}$	Monodromy locus: $\overline{c}_{21} = 0$. Other special loci: $\overline{c}_{31} = 0$ $\overline{c}_{21}' = 0$	$\begin{pmatrix} \omega^{a+2} & * & * \\ 0 & \mathrm{ind}_{G_{\mathbb{Q}_p}}^{G_{\mathbb{Q}_p}} \omega_2^{b+1+pc} \end{pmatrix}$	$F(a, b, c), F(a, c, b - p + 1)$
$\alpha\beta\alpha$	$\begin{pmatrix} 0 & 0 & v \\ 0 & v\overline{c}_{22}^* & v\overline{c}_{23} \\ v\overline{c}_{31}^* & v\overline{c}_{32} & v\overline{c}_{33}' \end{pmatrix}$ Shadow equation: $(a - b)\overline{c}_{23}\overline{c}_{32} - (a - c)\overline{c}_{22}^*\overline{c}_{33}' = 0$.	$\overline{c}_{33}' \neq 0$ $\overline{c}_{33}'\overline{c}_{22}^* \neq \overline{c}_{23}\overline{c}_{32}$	$\begin{pmatrix} \omega^{a+1} & * & * \\ 0 & \omega^{b+1} & * \\ 0 & 0 & \omega^{c+1} \end{pmatrix}$	$F(a - 1, b, c + 1);$ Shadow: $F(c + p - 1, b, a - p + 1)$
		$\overline{c}_{33}' \neq 0$ $\overline{c}_{33}'\overline{c}_{22}^* = \overline{c}_{23}\overline{c}_{32}$	$\begin{pmatrix} \mathrm{ind}_{G_{\mathbb{Q}_p}}^{G_{\mathbb{Q}_p}} \omega_2^{(a+1)+p(b+1)} & * \\ 0 & \omega^{c+1} \end{pmatrix}$	
		$\overline{c}_{33}' = 0$ $\overline{c}_{23}\overline{c}_{32} \neq 0$	$\begin{pmatrix} \omega^{a+1} & * & * \\ 0 & \mathrm{ind}_{G_{\mathbb{Q}_p}}^{G_{\mathbb{Q}_p}} \omega_2^{(b+1)+p(c+1)} \end{pmatrix}$	
		$\overline{c}_{33}' = 0$ $\overline{c}_{23} = 0$ $\overline{c}_{32} \neq 0$	$\begin{pmatrix} \mathrm{ind}_{G_{\mathbb{Q}_p}}^{G_{\mathbb{Q}_p}} (\omega_2^{(a+1)+p(c+1)}) & * \\ 0 & \omega^{b+1} \end{pmatrix}$	
		$\overline{c}_{33}' = 0$ $\overline{c}_{32} = 0$ $\overline{c}_{23} \neq 0$	$\begin{pmatrix} \mathrm{ind}_{G_{\mathbb{Q}_p}}^{G_{\mathbb{Q}_p}} (\omega_2^{(a+1)+p(c+1)}) & 0 \\ * & \omega^{b+1} \end{pmatrix}$	

$\alpha\beta$	$\begin{pmatrix} 0 & 0 & v \\ v\bar{c}_{21}^* & 0 & v\bar{c}_{23}' \\ 0 & v\bar{c}_{32}^* & v\bar{c}_{33}' \end{pmatrix}$	$\bar{c}_{33}'\bar{c}_{23}' \neq 0$	$\begin{pmatrix} \omega^{a+1} & * & 0 \\ 0 & \omega^{b+1} & * \\ 0 & 0 & \omega^{c+1} \end{pmatrix}$	$F(a-1, b, c+1),$ $F(a, c, b-p+1).$
		$\bar{c}_{33}' \neq 0$ $\bar{c}_{23}' = 0$	$\begin{pmatrix} \text{ind}_{G_{\mathbb{Q}_p}}^{G_{\mathbb{Q}_p}} \omega_2^{(a+1)+p(b+1)} & * \\ 0 & \omega^{c+1} \end{pmatrix}$	$F(a-1, b, c+1),$ $F(a, c, b-p+1).$
		$\bar{c}_{33}' = 0$ $\bar{c}_{23}' \neq 0$	$\begin{pmatrix} \omega^{a+1} & * \\ 0 & \text{ind}_{G_{\mathbb{Q}_p}}^{G_{\mathbb{Q}_p}} \omega_2^{(b+1)+p(c+1)} \end{pmatrix}$	$F(a-1, b, c+1),$ $F(a, c, b-p+1),$ $F(c+p-1, a, b);$ Shadow: $F(c+p-1, b, a-p+1)$
		$\bar{c}_{33}' = 0$ $\bar{c}_{23}' = 0$	$\omega \otimes \text{ind}_{G_{\mathbb{Q}_p}}^{G_{\mathbb{Q}_p}} \omega_3^{c+pb+p^2a}$	$F(a-1, b, c+1),$ $F(a, c, b-p+1),$ $F(c+p-1, a, b);$ Shadow: $F(c+p-1, b, a-p+1).$
$\beta\alpha$	$\begin{pmatrix} 0 & v & 0 \\ 0 & 0 & v\bar{c}_{23}^* \\ v\bar{c}_{31}^* & v\bar{c}_{32} & v\bar{c}_{33}' \end{pmatrix}$	$\bar{c}_{33}'\bar{c}_{32} \neq 0$	$\begin{pmatrix} \omega^{a+1} & * & * \\ 0 & \omega^{b+1} & * \\ 0 & 0 & \omega^{c+1} \end{pmatrix}$	$F(a-1, b, c+1),$ $F(b+p-1, a, c).$
		$\bar{c}_{33}' = 0$ $\bar{c}_{32} \neq 0$	$\begin{pmatrix} \omega^{a+1} & * \\ 0 & \text{ind}_{G_{\mathbb{Q}_p}}^{G_{\mathbb{Q}_p}} \omega_2^{(b+1)+p(c+1)} \end{pmatrix}$	$F(a-1, b, c+1),$ $F(b+p-1, a, c).$
		$\bar{c}_{33}' \neq 0$ $\bar{c}_{32} = 0$	$\begin{pmatrix} \text{ind}_{G_{\mathbb{Q}_p}}^{G_{\mathbb{Q}_p}} \omega_2^{(a+1)+p(b+1)} & * \\ 0 & \omega^{c+1} \end{pmatrix},$	$F(a-1, b, c+1),$ $F(b+p-1, a, c),$ $F(b, c, a-p+1),$ Shadow: $F(c+p-1, b, a-p+1)$
		$\bar{c}_{33}' = 0$ $\bar{c}_{32} = 0$	$\omega \otimes \left(\text{ind}_{G_{\mathbb{Q}_p}}^{G_{\mathbb{Q}_p}} \omega_3^{a+pb+p^2c} \right)$	$F(a-1, b, c+1),$ $F(b+p-1, a, c),$ $F(b, c, a-p+1),$ Shadow: $F(c+p-1, b, a-p+1).$

α	$\begin{pmatrix} 0 & v & 0 \\ v\bar{c}_{12}^* & v\bar{c}_{22}' & 0 \\ 0 & 0 & v\bar{c}_{33}^* \end{pmatrix}$	$\bar{c}_{22}' \neq 0$	$\begin{pmatrix} \omega^{a+1} & * & 0 \\ 0 & \omega^{b+1} & 0 \\ 0 & 0 & \omega^{c+1} \end{pmatrix}$	$F(a-1, b, c+1)$ $F(c+p-2, a, b+1)$ $F(a, c, b-p+1)$ $F(c+p-1, b, a-p+1)$ $F(b+p-1, a, c)$
		$\bar{c}_{22}' = 0$	$\omega \otimes \left(\omega^c \oplus \operatorname{ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{b+pa} \right)$	$F(a-1, b, c+1)$ $F(c+p-2, a, b+1)$ $F(a, c, b-p+1)$ $F(c+p-1, b, a-p+1)$ $F(b+p-1, a, c)$ $F(b, c, a-p+1)$
id	$\begin{pmatrix} v\bar{c}_{11}^* & 0 & 0 \\ 0 & v\bar{c}_{22}^* & 0 \\ 0 & 0 & v \end{pmatrix}$		$\omega^{a+1} \oplus \omega^{b+1} \oplus \omega^{c+1}$	$F(c+p-2, a, b+1),$ $F(a-1, b, c+1),$ $F(b-1, c, a-p+2),$ $F(p-1+b, a, c),$ $F(c+p-1, b, a-p+1),$ $F(a, c, b-p+1)$

TABLE 9. Serre weights for semisimple $\bar{\rho}$

$\bar{\rho}$	Obvious	Shadow
$(\omega^a \oplus \omega^b \oplus \omega^c) \otimes \omega$	$F(a-1, b, c+1),$ $F(b-1, c, a-p+2),$ $F(c+p-2, a, b+1),$ $F(a-1, c, b-p+2),$ $F(b+p-2, a, c+1),$ $F(c+p-2, b, a-p+2)$	$F(c+p-1, b, a-p+1),$ $F(b+p-1, a, c),$ $F(a, c, b-p+1)$
$(\omega^a \oplus \text{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{b+pc}) \otimes \omega$	$F(a-1, b, c+1),$ $F(b-1, c, a-p+2),$ $F(c+p-1, a, b),$ $F(a-1, c+1, b-p+1),$ $F(c+p-1, b-1, a-p+2),$ $F(b+p-1, a, c)$	$F(c+p-1, b, a-p+1),$ $F(a, c, b-p+1),$ $F(b+p-2, a, c+1)$
$(\text{Ind}_{G_{\mathbb{Q}_{p^3}}}^{G_{\mathbb{Q}_p}} \omega_3^{a+pb+p^2c}) \otimes \omega$	$F(a-1, b, c+1),$ $F(c+p-1, a-1, b+1),$ $F(b+p-1, c, a-p+1),$ $F(a-1, c+1, b-p+1),$ $F(c+p-1, b+1, a-p),$ $F(b+p-1, a, c)$	$F(c+p-1, b, a-p+1),$ $F(b+p-1, a-1, c+1),$ $F(a-1, c, b-p+2)$
$(\omega^b \oplus \text{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{a+pc}) \otimes \omega$	$F(a-1, b, c+1),$ $F(b-1, c+1, a-p+1),$ $F(c+p-1, a-1, b+1),$ $F(a-1, c, b-p+2),$ $F(c+p, b, a-p),$ $F(b+p-2, a, c+1)$	$F(c+p-1, b, a-p+1),$ $F(b+p-1, a-1, c+1),$ $F(a-1, c+1, b-p+1)$

The triple $(a, b, c) \in \mathbb{Z}^3$ verifies $0 < a - b$, $b - c < p - 1$ and $a - c < p - 1$. The table is deduced from [Her09], Lemma 7.6 and Proposition 6.28; alternatively the obvious weights can be deduced from [BLGG], Lemma 5.1.2.

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